Display
D
UNIVERSITY
OF MICHIGAN
NOV 13 1956
MATH. ECON.
LIBRARY

AMERICAN JOURNAL OF MATHEMATICS

FOUNDED BY THE JOHNS HOPKINS UNIVERSITY

EDITED BY

ANDRÉ WEIL
UNIVERSITY OF CHICAGO

AUREL WINTNER
THE JOHNS HOPKINS UNIVERSITY

WITH THE COÖPERATION OF

S. S. CHERN
C. CHEVALLEY
W. L. CHOW
J. A. DIEUDONNÉ
A. M. GLEASON
HARISH-CHANDRA
P. HARTMAN
G. P. HOCHSCHILD
I. KAPLANSKY
E. R. KOLCHIN
W. S MASSEY
D. C. SPENCER
A. D. WALLACE

PUBLISHED UNDER THE JOINT AUSPICES OF

THE JOHNS HOPKINS UNIVERSITY
AND
THE AMERICAN MATHEMATICAL SOCIETY

Volume LXXVIII, Number 4
OCTOBER, 1956

THE JOHNS HOPKINS PRESS
BALTIMORE 18, MARYLAND
U. S. A.

CONTENTS

	PAGE
A qualitative characterization of Blaschke products in a half-plane. By EDWIN J. AKUTOWICZ,	677
Solutions of some problems of division. Part III. By LEON EHRENPREIS,	685
Characteristic linear systems of complete continuous systems. By K. KODAIRA,	716
Fibre systems of Jacobian varieties. (II). By JUN-ICHI IGUSA,	745
Simultaneous resolution for algebraic surfaces. By SHREERAM ABHYANKAR,	761
Some algebraic number theory estimates based on the Dedekind eta-function. By HARVEY COHN,	791
Finite Fano planes. By ANDREW M. GLEASON,	797
Contribution to the Picard-Vessiot theory of homogeneous linear differential equations. By A. SEIDENBERG,	808
Cauchy's stable distributions and an "explicit formula" of Mellin. By AUREL WINTNER,	819
The decomposition theorem for V-manifolds. By WALTER L. BAILY, JR.,	862
Riemann metrics associated with convex bodies and normed spaces. By DETLEF LAUGWITZ and EDGAR R. LORCH,	889
Addenda to the paper on Bôcher's theorem. By AUREL WINTNER,	895
Correspondence,	898
Corrections to "Engel rings and a result of Herstein and Kaplansky." By M. P. DRAZIN,	899
Errata,	900

The AMERICAN JOURNAL OF MATHEMATICS appears four times yearly.

The subscription price of the JOURNAL is $8.50 in the U. S.; $8.75 in Canada; and $9.00 in other foreign countries. The price of single numbers is $2.50.

Manuscipts intended for publication in the JOURNAL should be sent to Professor AUREL WINTNER, The Johns Hopkins University, Baltimore 18, Md.

Subscriptions to the JOURNAL and all business communications should be sent to THE JOHNS HOPKINS PRESS, BALTIMORE 18, MARYLAND, U. S. A.

THE JOHNS HOPKINS PRESS supplies to the authors 100 free reprints of every article appearing in the AMERICAN JOURNAL OF MATHEMATICS. On the other hand, neither THE JOHNS HOPKINS PRESS nor the AMERICAN JOURNAL OF MATHEMATICS can accept orders for additional reprints. Authors interested in securing more than 100 reprints are advised to make arrangements directly with the printers, J. H. FURST CO., 20 HOPKINS PLACE, BALTIMORE 1, MARYLAND.

The typescripts submitted can be in English, French, German or Italian and should be prepared in accordance with the instructions listed on the inside back cover of this issue.

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918.

PRINTED IN THE UNITED STATES OF AMERICA
BY J. H. FURST COMPANY, BALTIMORE, MARYLAND

A QUALITATIVE CHARACTERIZATION OF BLASCHKE PRODUCTS IN A HALF-PLANE.*[1]

By EDWIN J. AKUTOWICZ.

---

1. Introduction. Let $a_1, a_2, \cdots$ be a sequence of complex numbers with $\operatorname{Im} a_\nu > 0$ for all ν, and such that

$$\sum_{\nu=1}^{\infty} \operatorname{Im} a_\nu/(1+|a_\nu|^2) < \infty. \tag{1}$$

This condition guarantees that the Blaschke product $b(z)$ with the a_ν as its set of zeros converges uniformly in any bounded closed set contained in the half-plane $\operatorname{Im} z \geqq a$, where a is any positive number, and independently of the order of the factors:

$$b(z) = ((z-i)/(z+i))^n \prod_{\nu=1}^{\infty} (|a_\nu - i|/(a_\nu - i))(|a_\nu + i|/(a_\nu + i)) \times ((z-a_\nu)/(z-\bar{a}_\nu)), \tag{2}$$

where n is a non-negative integer. *We make the fixed assumption in Section* 1 *and Section* 2 *that* $\operatorname{Im} a_\nu > 0$ *and that* (1) *holds.*

It is familiar that $|b(x+iy)| < 1$ for $y > 0$, and that $|b(x+iy)| \to 1$ for almost all x, $-\infty < x < \infty$, as $y \to 0+$. We shall prove that

$$\int_{-\infty}^{\infty} \log|b(z)|/(1+x^2)dx \to 0 \text{ as } y \to 0+. \tag{3}$$

(We write $z = x+iy$). Furthermore, we shall prove in Section 3 that any function $F(z)$, holomorphic in the half-plane $y > 0$, satisfying the conditions $|F(z)| < 1$ and

$$\int_{-\infty}^{\infty} \log|F(z)|/(1+x^2)dx \to 0 \text{ as } y \to 0+,$$

is necessarily of the form $F(z) = e^{ikz+ic}b(z)$, where k, c are real constants, $k \geqq 0$, and $b(z)$ is the Blaschke product formed with the zeros of $F(z)$. If one also assumes $(1/r)\log|F(re^{i\theta})| \to 0$ as $r \to \infty$, for some θ, $0 < \theta < \pi$,

---

* Received April 22, 1955; revised May 14, 1956.

[1] This research was supported jointly by the Army, Navy, and Air Force under contract with the Lincoln Laboratory, Massachusetts Institute of Technology.

then $k=0$. This characterization of $b(z)$ is thus in terms of a combination of general properties which have no explicit reference to the formula (2).

2. Proof of Necessity. The product $b(z)$, given by (2), is convergent for $y>0$ and we can choose a positive real number λ, which will be fixed once and for all, such that $b(i\lambda)\neq 0$. Then $b(iy)$ does not vanish for y in some closed interval $\mathfrak{J}$ centered at λ. We may take $\mathfrak{J}$ such that the length of $\mathfrak{J}$ is smaller than λ. Let ϵ be positive and such that $\lambda-\epsilon$ belongs to $\mathfrak{J}$ (so that $\epsilon\leqq\lambda/2$), and map $\operatorname{Im} z>0$ to $|w|<1$ by $w=(z-i(\lambda-\epsilon))/(z+i(\lambda-\epsilon))$. Then the horizontal line $y=\epsilon$ corresponds to the circle C_ϵ with center $(\epsilon/\lambda, 0)$ and radius $(\lambda-\epsilon)/\lambda$. The line element $|dw|$ on C_ϵ is related to the line element dx on $y=\epsilon$ by the equation $|dw|=\{2(\lambda-\epsilon)/(x^2+\lambda^2)\}dx$. Now

$$0<\min(1,\lambda^2)\leqq(\lambda^2+x^2)/(1+x^2)\leqq\max(1,\lambda^2)\quad\text{for}\quad-\infty<x<\infty$$

implies that, as $y\to 0+$,

$$\int_{-\infty}^{\infty}\log|b(z)|/(1+x^2)\,dx\to 0\quad\text{if and only if}\quad\int_{-\infty}^{\infty}\log|b(z)|/(x^2+\lambda^2)\,dx\to 0.$$

Therefore for $\operatorname{Im} z=y=\epsilon$, we have [2] $\int_{-\infty}^{\infty}\log|b(z)|/(x^2+\lambda^2)\,dx$

$$=\int_{-\infty}^{\infty}(x^2+\lambda^2)^{-1}\log\prod_\nu|(z-a_\nu)/(z-\bar a_\nu)|\,dx$$

$$=\sum_\nu\int_{-\infty}^{\infty}(x^2+\lambda^2)^{-1}\log|(z-a_\nu)/(z-\bar a_\nu)|\,dx$$

$$=(2(\lambda-\epsilon))^{-1}\sum_\nu\int_{C_\epsilon}\log|\{[i(\lambda-\epsilon)(1+w)/(1-w)]-a_\nu\}$$
$$/\{[i(\lambda-\epsilon)(1+w)/(1-w)]-\bar a_\nu\}|\,|dw|$$

$$=(2(\lambda-\epsilon))^{-1}\sum_\nu\{(2\pi(\lambda-\epsilon)/\lambda)\log|(i(\lambda-\epsilon)+a_\nu)/(i(\lambda-\epsilon)-a_\nu)|$$
$$+\int_{C_\epsilon}\log|(w-k_\nu)/(w-k'_\nu)|\,|dw|\},$$

where

$$k_\nu=-(i(\lambda-\epsilon)-a_\nu)/(i(\lambda-\epsilon)+a_\nu),$$
$$k'_\nu=-(i(\lambda-\epsilon)-\bar a_\nu)/(i(\lambda-\epsilon)+\bar a_\nu).$$

Then $|k_\nu|<1$ and $|k'_\nu|>1$, and if k_ν lies in the interior or on the boundary of C_ϵ we can use Jensen's formula [5] to obtain

[2] We take $n=0$ in (2), which is clearly permissible.

$$\int_{C_\epsilon} \log|(w-k_\nu)/(w-k'_\nu)|\,|\,dw\,|$$
$$= ((\lambda-\epsilon)/\lambda)\int_{|s|=(\lambda-\epsilon)/\lambda} \log|\{s-(k_\nu-(\epsilon/\lambda))\}/\{s-(k'_\nu-(\epsilon/\lambda))\}|\,d\arg s$$
$$= (2\pi(\lambda-\epsilon)/\lambda)\log|(i(\lambda-\epsilon)-a_\nu)/(i(\lambda+\epsilon)+a_\nu)|.$$

If k_ν lies outside C_ϵ, $\log|(w-k_\nu)/(w-k'_\nu)|$ is harmonic in a neighborhood of the closed disk bounded by C_ϵ and the Gauss mean value theorem yields

$$\int_{C_\epsilon} \log|(w-k_\nu)/(w-k'_\nu)|\,|\,dw\,| = 2\pi((\lambda-\epsilon)/\lambda)\log|((\epsilon/\lambda)-k_\nu)/((\epsilon/\lambda)-k'_\nu)|$$
$$= 2\pi((\lambda-\epsilon)/\lambda)\log|(i(\lambda+\epsilon)-a_\nu)(i(\lambda-\epsilon)-a_\nu)$$
$$/(i(\lambda+\epsilon)+a_\nu)(i(\lambda-\epsilon)+a_\nu)|.$$

Hence

$$\int_{-\infty}^{\infty}(x^2+\lambda^2)^{-1}\log|b(z)|\,dx = (\pi/\lambda)\sum_\nu\Bigg(\log|(i(\lambda-\epsilon)+a_\nu)/(i(\lambda-\epsilon)-a_\nu)|$$
$$+\begin{cases}\log|(i(\lambda-\epsilon)-a_\nu)/(i(\lambda+\epsilon)+a_\nu)| & \text{if } \operatorname{Im} a_\nu \geqq \epsilon\\ \log|(i(\lambda+\epsilon)-a_\nu)(i(\lambda-\epsilon)-a_\nu)/(i(\lambda+\epsilon)+a_\nu)(i(\lambda-\epsilon)+a_\nu)| & \text{if } \operatorname{Im} a_\nu < \epsilon\end{cases}\Bigg)$$

$$(4)\qquad = (\pi/\lambda)\sum_{\operatorname{Im} a_\nu \geqq \epsilon}\log|(i(\lambda-\epsilon)+a_\nu)/(i(\lambda+\epsilon)+a_\nu)|$$
$$+(\pi/\lambda)\sum_{\operatorname{Im} a_\nu < \epsilon}\log|(i(\lambda+\epsilon)-a_\nu)/(i(\lambda+\epsilon)+a_\nu)|.$$

In order to prove that the first sum in (4), which we shall denote by Δ, tends to 0 as $\epsilon\to 0$ we write $b(z)=b_1(z)b_2(z)b_3(z)$, where the zeros of $b_1(z)$ lie in the rectangle $0<y<1$, $|x|<P$, those of $b_2(z)$ in the half-plane $y\geqq 1$, and those of $b_3(z)$ in the two infinite strips $0<y<1$, $|x|\geqq P$. Here P is at our disposal.

The zeros of the product $b_1(z)$ can be enumerated so that

$$(5)\qquad y_1\geqq y_2\geqq y_3\geqq\cdots,\ \lim y_\nu = 0,$$

where $a_\nu = x_\nu + iy_\nu$, $|x_\nu|<P$, $0<y_\nu<1$.
Then the first sum in (4) for $b=b_1$ contains a finite number of summands, and so is certainly finite. We now prove that it tends to 0 as $\epsilon\to 0+$. Essential use is made of (5). We have

$$\Delta_1 = \sum_{y_\nu\geqq\epsilon}\log|(i(\lambda-\epsilon)+a_\nu)/(i(\lambda+\epsilon)+a_\nu)|.$$

Then, for $\epsilon = y_q$, q large,

$$2\Delta_1 = \sum_{\nu=1}^{q} \log\left(\{(\lambda + y_\nu - y_q)^2 + x_\nu^2\}/\{(\lambda + y_\nu + y_q)^2 + x_\nu^2\}\right)$$

$$= \sum_{\nu=1}^{q} \{(4y_q(\lambda + y_\nu)/(\lambda + y_\nu + y_q)^2 + x_\nu^2) \text{ times a uniformly bounded factor}\},$$

by the mean value theorem, plus the fact that $(i(\lambda - \epsilon) + a_\nu)/(i(\lambda + \epsilon) + a_\nu)$ is bounded away from 0, uniformly for all ν and $\epsilon \leqq \lambda/2$. Therefore,

$$|2\Delta_1| \leqq My_q \sum_{\nu=1}^{q} (\lambda^2 + y_\nu^2 + x_\nu^2)^{-1} + My_q \sum_{\nu=1}^{q} y_\nu(\lambda^2 + y_\nu^2 + x_\nu^2)^{-1}.$$

The second sum obviously tends to 0 as $q \to \infty$, and for $q > N$

$$y_q \sum_{\nu=1}^{q} (\lambda^2 + y_\nu^2 + x_\nu^2)^{-1} < y_q \sum_{\nu=1}^{N} (\lambda^2 + y_\nu^2 + x_\nu^2)^{-1} + \sum_{\nu=N+1}^{\infty} y_\nu(\lambda^2 + y_\nu^2 + x_\nu^2)^{-1}.$$

For N sufficiently large the last sum in the preceding line is arbitrarily small, and the rest tends to 0 since $y_q \to 0$ as $q \to \infty$. Therefore $\Delta_1 \to 0$ for any $P > 0$.

We now consider $b_2(z)$, and take $\epsilon < \min(1, \lambda/2)$. For this product

$$\begin{aligned} 0 < -2\Delta_2 &= \sum_{y_\nu \geqq 1} \log |(i(\lambda + \epsilon) + a_\nu)/(i(\lambda - \epsilon) + a_\nu)|^2 \\ &= \sum_{y_\nu \geqq 1} \log\{1 + 4\epsilon[(y_\nu + \lambda)/(x_\nu^2 + (y_\nu + \lambda - \epsilon)^2]\} \\ &< 4\epsilon \sum_{y_\nu \geqq 1} (y_\nu + \lambda)/(x_\nu^2 + (y_\nu + (\lambda/2))^2) \\ &< 4\epsilon(\lambda + 2) \sum_{y_\nu \geqq 1} y_\nu/(x_\nu^2 + y_\nu^2). \end{aligned}$$

Hence $\Delta_2 \to 0$ as $\epsilon \to 0$, since the last sum is finite.

For $b_3(z)$ with $P = 1$ we have

$$\sum_{0 < y_\nu < 1,\ |x_\nu| \geqq 1} y_\nu/(x_\nu^2 + y_\nu^2) < \infty,$$

and hence if $\epsilon' > 0$ we can fix P at a sufficiently large value, *independent of* ϵ, so that

$$\sum_{0 < y_\nu < 1,\ |x_\nu| \geqq P} y_\nu/(x_\nu^2 + y_\nu^2) < \epsilon'.$$

Then, writing $\Sigma' = \sum_{\epsilon \leqq y_\nu < 1,\ |x_\nu| \geqq P}$, it follows that

$$\begin{aligned} 0 < -2\Delta_3 &< 4\epsilon \Sigma'(y_\nu + \lambda)/\{x_\nu^2 + (y_\nu + (\lambda/2))^2\} \\ &\leqq 4\epsilon(1 + \lambda) \Sigma' 1/(x_\nu^2 + y_\nu^2) \\ &\leqq 4(1 + \lambda) \Sigma' y_\nu/(x_\nu^2 + y_\nu^2) < 4\epsilon'(1 + \lambda), \end{aligned}$$

and this proves that Δ_3 is arbitrarily small. Therefore $\Delta \to 0$ as $\epsilon \to 0$ for any Blaschke product, since $\Delta = \Delta_1 + \Delta_2 + \Delta_3$.

Turning now again to (4), it is easy to see that $\sum_{\mathrm{Im}\, a_\nu < \epsilon}$ tends to 0 as $\epsilon \to 0+$ for any Blaschke product. (Here we again use the fact that $b(i\lambda) \neq 0$.) Thus (3) holds for any Blaschke product $b(z)$. Hence we have proved

THEOREM 1. *Let $b(z)$ be a Blaschke product in the upper half-plane,* $\mathrm{Im}\, z = \mathrm{Im}\,(x+iy) = y > 0$. *Then*

$$\int_{-\infty}^{\infty} (1+x^2)^{-1} \log |b(z)|\, dx \to 0 \quad \text{as} \quad y \to 0+.$$

COROLLARY. *For any finite real numbers l_1 and l_2, $l_1 < l_2$, and any Blaschke product $b(z)$,*

$$\int_{l_1}^{l_2} \log |b(x+iy)|\, dx \to 0 \quad \text{as} \quad y \to 0+.$$

This has been established by W. Kryloff [2] by a rather complicated argument. It is a trivial consequence of Theorem 1 and the inequality

$$(1 + \{\max(|l_1|, |l_2|)\}^2) \int_{-\infty}^{\infty} (1+x^2)^{-1} \log |b(z)|\, dx \leqq \int_{l_1}^{l_2} \log |b(z)|\, dx < 0.$$

3. Proof of sufficiency. In this section we establish the following uniqueness theorem.

THEOREM 2. *Let $F(z)$ be holomorphic for* $\mathrm{Im}\, z = \mathrm{Im}(x+iy) = y > 0$, *and such that $|F(z)| < 1$ and*

$$\int_{-\infty}^{\infty} (1+x^2)^{-1} \log |F(z)|\, dx \to 0 \quad \text{as} \quad y \to 0+. \tag{6}$$

Then $F(z) = e^{ikz+ic} b(z)$, where k and c are real constants, $k \geqq 0$, and $b(z)$ is the Blaschke product with the zeros of $F(z)$ as its set of zeros.

In the first place we claim that the Blaschke product $b(z)$ formed with the zeros of $F(z)$ is convergent. For we can put $f(w) = F(i(1+w)/(1-w))$ to define a bounded holomorphic function $f(w)$ in $|w| < 1$, $w = \rho e^{i\theta}$. The integral of $\log |f(\rho e^{i\theta})|$ over $0 \leqq \theta < 2\pi$ does not decrease when ρ increases, and since $f(w)$ is bounded, it must increase to a finite limit as $\rho \to 1-$. Therefore, by a theorem of A. Ostrowski [3] [3] the Blaschke product [4] in the

[3] This result of Ostrowski is much stronger than what is necessary for our purposes.

[4] This is given by $w^n \prod_\nu (|w_\nu|/w_\nu)((w_\nu - w)/(1 - \bar{w}_\nu w))$, $\{w_\nu\}$ being the set of zeros of f which are different from 0, and n a non-negative integer.

unit circle, formed with the zeros of $f(w)$, is convergent in $|w|<1$, and this is equivalent to $b(z)$ formed with the zeros of $F(z)$ being convergent in $y>0$. Let us take the zeros of $F(z)$ in any enumeration: $a_1, a_2, \cdots$. Let $b_m(z)$ denote the m-th partial product of $b(z)$. In modulus, the factors of $b_m(z)$ are $|(z-a_\nu)/(z-\bar{a}_\nu)|$, $\nu=1,2,\cdots,m$, and if $\eta>0$ is given, there exists a positive number $\epsilon_0(m,\eta)$ such that

$$|(x+i\epsilon-a_\nu)/(x+i\epsilon-\bar{a}_\nu)|>1-(\eta/m), \text{ for } 0<\epsilon<\epsilon_0(m,\eta), \quad -\infty<x<\infty, \ \nu=1,2,\cdots,m.$$

Therefore, $|b_m(x+i\epsilon)|>(1-(\eta/m))^m>(1-\eta)$, and

$$|F(x+i\epsilon)/b_m(x+i\epsilon)|<(1-\eta)^{-1} \text{ for } 0<\epsilon<\epsilon_0(m,\eta), \ -\infty<x<\infty.$$

By the maximum principle,

$$|F(z)/b_m(z)|<(1-\eta)^{-1} \text{ for } 0<\epsilon<\epsilon_0(m,\eta), \ \operatorname{Im} z\geqq\epsilon.$$

This implies, successively, that

$$|F(z)/b_m(z)|<(1-\eta)^{-1}, \ |F(z)/b(z)|\leqq(1-\eta)^{-1}, \ |F(z)/b(z)|\leqq 1, \quad \text{for } \operatorname{Im} z>0.$$

Therefore, $\phi(z)\equiv F(z)/b(z)$ is holomorphic, bounded in modulus by 1, and nonvanishing in $y>0$. By (6) and Theorem 1,

$$(7) \qquad \int_{-\infty}^{\infty}(1+x^2)^{-1}\log|\phi(x+iy)|\,dx$$

$$=\int_{-\infty}^{\infty}(1+x^2)^{-1}\log|F(x+iy)|\,dx-\int_{-\infty}^{\infty}(1+x^2)^{-1}\log|b(x+iy)|\,dx\to 0$$

as $y\to 0+$. It is easy to see that $\log|\phi(x+iy)|$ can be written in the form

$$(8) \quad \log|\phi(x+iy)|=-ky-\frac{1}{\pi}\int_{-\infty}^{\infty}y(1+t^2)/(y^2+(x-t)^2)\,dE(t), \quad y>0, k\geqq 0,$$

where $E(t)$ is a bounded increasing function on the closed infinite interval $[-\infty,\infty]$. We normalize $E(t)$ to be continuous from the right. This follows from a theorem of Herglotz [4] for $|w|<1$ by the transformation $z=i(1-w)/(1+w)$. The term $-ky$ arises by subtracting a possible jump at infinity, so that $E(t)\to E(\infty)$ as $t\to\infty$, and $E(t)\to E(-\infty)$ as $t\to-\infty$. Let $u(x,y)$ denote the non-negative harmonic function given by the integral on the right hand side of (8). Then, by (7),

$$\int_{-\infty}^{\infty} u(x,y)/(1+x^2)\,dx \to 0 \text{ as } y \to 0+.$$

But

$$\int_{-\infty}^{\infty} u(x,y)/(1+x^2)dx = \tfrac{1}{\pi}\int_{-\infty}^{\infty}(1+x^2)^{-1}dx \int_{-\infty}^{\infty} y(1+t^2)(y^2+(x-t)^2)^{-1}dE(t)$$

$$= \tfrac{1}{\pi}\int_{-\infty}^{\infty}(1+t^2)dE(t)\int_{-\infty}^{\infty}\{(1+x^2)(y^2+(x-t)^2)\}^{-1}y\,dx$$

$$= \int_{-\infty}^{\infty}(1+y)(1+t^2)((1+y)^2+t^2)^{-1}\,dE(t) = \pi u(0,1+y),^{5}$$

where the interchange of the order of integration is valid on account of positivity. It follows that

$$0 = u(0,1) = \tfrac{1}{\pi}\int_{-\infty}^{\infty} dE(t) = \tfrac{1}{\pi}(E(\infty)-E(-\infty)).$$

Therefore, $E(t) \equiv$ constant, and $\log|\phi(x+iy)| = -ky$, $\phi(z) = e^{ikz+ic}$, where c is a real constant. This proves that $F(z) = e^{ikz+ic}b(z)$.

Remark 1. It is known [1] that for any $\theta_0 > 0$, $\log|b(z)|/|z| \to 0$ uniformly for $\theta_0 < \arg z < \pi - \theta_0$ as $|z| \to \infty$ outside a certain set $\Delta = \Delta(\theta_0)$ of finite logarithmic length.[6] Therefore, if in addition to the conditions stated in Theorem 2, one imposes the further requirement that for some $\theta(0 < \theta < \pi)$, $(\log|F(re^{i\theta})|)/r \to 0$ as $r \to \infty$, then $k = 0$ and $F(z) = e^{ic}b(z)$.

Remark 2. *A propos* the unit circle there is the following well-known and easily proved result:

Any holomorphic function $f(w)$ in $|w| < 1$ satisfying the conditions

(i) $|f(w)| < 1$,

(ii) $$\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta \to 0, \text{ as } r \to 1-,$$

is of the form $f(w) = e^{ic}b(w)$, where c is a real number and $b(w)$ is the Blaschke product formed with the zeros of $f(w)$.

Our uniqueness theorem is less precise than this in that the factor e^{ikz} can occur. Of course, this difference originates from the fact that (ii) does not correspond to (6).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

[5] We use here the semi-group property of the kernel $c(x;\lambda,\mu) = \frac{1}{\pi}\lambda(\lambda^2+(x-\mu)^2)^{-1}$: $c(x;\lambda_1,\mu_1) * c(x;\lambda_2,\mu_2) = c(x;\lambda_1+\lambda_2,\mu_1+\mu_2)$, $*$ denoting convolution in $L(-\infty,\infty)$.

[6] The logarithmic length of a measurable subset Δ of $(0,\infty)$ is defined as $\int_\Delta dr/r$.

REFERENCES.

[1] R. P. Boas, Jr., *Entire Functions*, New York, 1954, p. 115.

[2] W. Kryloff, "On functions which are regular in a half-plane," *Recueil Mathematique*, vol. 6 (48) (1939), in particular pp. 113-116.

[3] A. Ostrowski, "Über die Bedeutung der Jensenschen Formel für einige Fragen der komplexen Funktionentheorie," *Acta Litterarum ac Scientiarum*, Szeged, 1923, pp. 80-81.

[4] M. H. Stone, *Linear Transformations in Hilbert Space*, American Mathematical Society Colloquium Publications, vol. XV (1932), pp. 570-571.

[5] E. C. Titchmarsh, *Theory of Functions*, 2nd ed., Oxford, 1939, p. 125.

SOLUTIONS OF SOME PROBLEMS OF DIVISION.*[1]

Part III. Division in the Spaces, $\mathcal{D}'$, $\mathcal{H}$, $\mathcal{Q}_A$, $\mathcal{O}$.

By LEON EHRENPREIS.

---

1. Introduction. The main division problem is the following: Let $\mathcal{D}$ be a differential operator with constant coefficients and let T be a distribution (see [12]); can we find a distribution S which satisfies

$$DS = T \; ? \tag{1}$$

If T is a distribution of finite order, that is, if T can be written as a finite sum of derivatives of continuous functions, then the existence of an S satisfying (1) was proven by the author (see [2]) and, independently, by B. Malgrange in his thesis (see [11]). In the present paper we shall give the complete solution to the division problem. Moreover, we shall extend our results to the case where D is a partial differential-difference operator with constant coefficients. This general result is apparently new even in the case of ordinary differential-difference equations.

The question naturally arises: Which spaces of distributions or functions have the property that $f \to Df$ maps the space onto itself? Our main result is that the space $\mathcal{D}'$ of all distributions has this property; previously we had shown that the space of distributions of finite order, and the space of indefinitely differentiable functions have this property.

For the space $\mathcal{H}$ of entire functions (see [4]) even more is true: For any $W \in \mathcal{H}'$, and any $f \in \mathcal{H}$, we can find a $g \in \mathcal{H}$ such that

$$W * g = f. \tag{2}$$

(This result was discovered independently by Malgrange in his thesis [11]). What other spaces have this property in addition to the space $\mathcal{H}$?

We shall see that the solution of the two problems mentioned above can be translated into problems concerning the Fourier transform of the dual of the space in question. For example, the Fourier transform of the space $\mathcal{D}$ is the space of all entire functions of exponential type which lie in the space

---

* Received August 28, 1955.

[1] Work supported by National Science Foundation Grant NSF5-G1010.

$\mathcal{S}$ of Schwartz (see [2], [12]), while the Fourier transform of the space $\mathcal{H}'$ is the space of all entire functions of exponential type. The question arises, what is the topology of these spaces of Fourier transforms? Once this topology is written down explicitly, it is not difficult to apply our previous methods (see [2], [3]) to show that $Dh \to h$ is a continuous linear map of $D\mathcal{D}$ into $\mathcal{D}$ ($D\mathcal{D}$ is the space of all Dh for $h \in \mathcal{D}$ with the topology induced by $\mathcal{D}$) and that $W * U \to U$ is a continuous linear map of $W * \mathcal{H}'$ into $\mathcal{H}'$. Once this is established, we can use the Hahn-Banach theorem to show that, for any $T \in \mathcal{D}'$ or for any $f \in \mathcal{H}$, equations (1) and (2) are solvable.

From the above, we see that there are two important steps in solving our main problems for a space A:

Problem 1. Characterize explicitly the Fourier transform of the dual A' of A and its topology.

Problem 2. Find a class of convolution maps $h \to W * h$ of $A \to A$ which have the property that $W * h \to h$ is a continuous map of $W * A'$ into A'.

Now, Problem 2 can usually be solved when Problem 1 can be solved and when the Fourier transform $\mathcal{A}'$ of A' is a space of analytic functions. There are essentially two cases:

Case 1. The space $\mathcal{A}'$ consists of functions defined only by regularity conditions or by growth conditions in the whole complex space. E. g. the space of all entire functions of exponential type, the space of all entire functions, the space of all entire functions of finite order. For these spaces we can use the minimum modulus results of [4] to show that, in most cases, if A' is a ring under convolution, then for every $W \in A'$, $W * h \to h$ is a continuous linear map of $W * A' \to A'$.

Case 2. The space $\mathcal{A}'$ consists of functions defined by growth conditions in the whole complex space and additional conditions imposed on the real subspace. E. g. the Fourier transform of $\mathcal{D}$ or $\mathcal{E}'$ (see [5]), or many of the spaces which arise in the theory of infinite derivatives (see [7]). For these spaces, in order to prove that $W * h \to h$ is a continuous linear map of $W * A' \to A'$, we have to know that the Fourier transform of W does not decrease too rapidly on the real subspace. That is why, for these spaces, we can usually prove the continuity only for W a differential-difference

operator, where we can use the theory of mean-periodic functions (see [3], [4]) to get lower bounds on the real subspace.[2]

For many of the spaces we encounter, the solution of Problem 1 can be obtained by methods which are inspired by results in my paper on the theory of infinite derivatives (see [7]). This is not the case with the space $\mathcal{D}$; the characterization of the topology of the Fourier transform $\boldsymbol{D}$ of $\mathcal{D}$ is extremely difficult because the space $\mathcal{D}$ cannot be defined by the methods of the theory of infinite derivatives. This is the reason why the solution of the division problem for $\mathcal{D}'$ was so difficult to obtain.

Some of the results of this paper regarding the spaces $\mathcal{D}$ and $\mathcal{H}$ were announced in [8]. In addition, we shall show how to solve the division problems for the spaces

1. $\mathcal{Q}_A$ of entire functions of order $\leqq A$;
2. $\mathcal{O}$ of formal power series;
3. $\mathcal{B}$ of convergent power series;
4. $\mathcal{E}$ of indefinitely differentiable functions.

Of course, our methods apply to many other spaces (see Section 7 below).

The notations of this paper will be that of Part I (see [2], Section 2) and the results of Parts I and II will be used here in an essential way (see [2], [3]).

2. Solution of the main division problem. As mentioned in the introduction, the first step in the solution of the division problem for $\mathcal{D}'$ is the characterization of the topology of $\boldsymbol{D}$. Now, the characterization of the topology of $\boldsymbol{D}_F$ (see [2]) (for $n=1$) was obtained by means of giving bounds on functions of $\boldsymbol{D}_F$ on a sequence of lines parallel to the real axis. That we cannot define the topology of $\boldsymbol{D}$ in terms of bounds on horizontal lines is a consequence of a Phragmén-Lindelöf theorem (see [2], Lemma 1, p. 887). For, this shows that, if $F \in \boldsymbol{D}$, then a bound on Z^jF on any horizontal line implies a bound on the real axis. Thus, if a neighborhood N of zero in $\boldsymbol{D}$ is to be defined in terms of bounds of Z^jF on horizontal lines, only a finite number of j can occur. (For otherwise, we could find a $G \in \boldsymbol{D}$ so that, for no $a > 0$ is $aG \in N$.) Thus, we can only get neighborhoods of zero in $\boldsymbol{D}_F$ by means of bounds on horizontal lines.

[2] We can even ask the question of finding all W for which $W * h \to \delta$ is a continuous linear map of A' into A'. In case $A = \mathcal{D}'$, this question has been solved by the author (see "Completely inversible operators," *Proceedings of the National Academy of Science*, vol. 41 (1955), pp. 945-946). The case $A = \mathcal{E}$ is also handled there.

Thus, if a "good" description of the topology of $\boldsymbol{D}$ is possible, it must be given in terms of bounds on something else. This "something else" which replaces horizontal lines is a sequence of curves:

THEOREM 1. *The topology of $\boldsymbol{D}$ can be described as follows: We choose first positive numbers c, η. Next, for each integer $l \geqq 0$, we choose*

(a) *Two non-negative integers a_l, d_l with $\{a_l\}$, $\{d_l\}$ strictly increasing to infinity, with $a_0 = d_0 = 0$, and $a_{l+1} \geqq a_l + c + 1$.*

(b) *A positive integer $b_l \geqq 5$ such that $\{b_l\}$ is monotonically increasing.*

We consider the set N of all $G \in \boldsymbol{D}$ which satisfy, for each l,

$$\max |\exp(3id_l p_k \cdot z) z_j^s G(z)| \leqq \eta, \text{ where } z \in \Gamma_l^k \tag{3}$$

for $s = 0, 1, 2, \cdots, b_l$, $j = 1, 2, \cdots, 2^n$, $k = 1, 2, \cdots, 2^n$, where Γ_0^k is, for each k, the set of z such that $|\mathfrak{A}(z_j)| \leqq c$ for all j, and Γ_l^k for $l > 0$ are defined as follows: For each $z \in C$, let γ_l be defined by $\gamma_0 = 0$ and, for $l > 0$,

$$\exp(d_l \gamma_l) = \begin{cases} |\zeta|^{b_l} & \text{for } |\zeta| \geqq 1 \\ 1 & \text{for } |\zeta| \leqq 1 \end{cases} \tag{4}$$

where $\zeta = (R(z_1), R(z_2), \cdots, R(z_n))$ and $|\zeta| = \sum |\zeta_j|$, that is,

$$\gamma_l = \begin{cases} b_l \log |\zeta| / d_l & \text{for } |\zeta| \geqq 1 \\ 0 & \text{for } |\zeta| \leqq 1. \end{cases} \tag{5}$$

Then Γ_l^k is the set consisting of all $z \in C^k$ for which

$$z_j = \zeta_j + i\gamma_l p_{kj} + i p_{kj} a_l + i\xi, \quad -c < \xi < c, \qquad j = 1, 2, \cdots, n. \tag{6}$$

ASSERTION. *The sets N described above form a fundamental system of neighborhoods of zero in $\boldsymbol{D}$.*

Proof. The sets N described above are obviously convex. Let $F \in \boldsymbol{D}$; then for some $\alpha > 0$, $F \in \boldsymbol{D}_\alpha$. Assume at first that l is chosen so large that $d_l \geqq \alpha$. Then, by a generalized Phragmén-Lindelöf theorem (see [2], Lemma 1, p. 887) for each k we know that $\exp(id_l p_k \cdot z) F(z)$ is bounded for $z \in C^k$ by the bound M of F on R. It follows that, whenever $d_l \geqq \alpha$, we have

$$\max_{z \in \Gamma_l^k} |\exp(3id_l p_k \cdot z) z_j^s F(z)| \leqq M \tag{7}$$

for $s = 0, 1, 2, \cdots, b_l$, $j = 1, 2, \cdots, n$, and $k = 1, 2, \cdots, 2^n$, because on Γ_l^k, $|\exp(id_l p_k \cdot z) z_j^s| \leqq 1$ for $s = 0, 1, 2, \cdots, b_l$ and for $j = 1, 2, \cdots, n$, as follows easily from the definitions.

Next we consider the finite number of l for which $d_l < \alpha$.

Let δ_l^k be the set of $z \in C^k$ of the form $z_j = \zeta_j + i\gamma_l p_{k_l} + i a_l p_{k_l}$ for all j; then for any $l \geqq 1$, on δ_l^k we have

$$|\exp(-id_l p_k \cdot z)| \leqq \exp(d_l a_l)\,(1 + \Sigma\,|z_j|^{nb_l}). \tag{8}$$

From (8) it follows that, on δ_l^k,

$$|\exp(-i\alpha p_k \cdot z)| \leqq q_l(1 + \Sigma\,|z_j|^{\beta_l}) \tag{9}$$

if β_l and q_l are chosen sufficiently large. Hence, for any $z \in \Gamma_l^k$, if z' denotes the point on δ_l^k such that $\mathcal{R}(z_j) = \mathcal{R}(z_j')$ for all j, we have by (9),

$$\begin{aligned}|\exp(-i\alpha p_k \cdot z)| &\leqq \exp(\alpha)\,|\exp(-i\alpha p_k \cdot z')| \\ &\leqq q_l(1 + \Sigma\,|z_j'|^{\beta_l})\exp(c\alpha) \leqq q_l'(1 + \Sigma\,|z_j|^{\beta_l}) \text{ for some } q_l' > 0.\end{aligned} \tag{10}$$

Now, let β_l' be an integer and let M' be a bound for $|z_j^{\beta_l'} F(z)|$, $|F(z)|$ on R for $j = 1, 2, \cdots, n$. By the Phragmén-Lindelöf theorem, M' is also a bound for $|\exp(i\alpha p_k \cdot z) z_j^{\beta_l'} F(z)|$ and $|\exp(i\alpha p_k \cdot z) F(z)|$ in C^k, *a fortiori* in Γ_l^k. Using (10) we deduce that, if β_l' is large enough, we can find an $a > 0$ so that aF satisfies (3) for this particular l. Since there are only a finite number of l for which $d_l < \alpha$, we can even choose a independent of l.

Summing up what we have done so far, we see:

(a) *For each $F \in \boldsymbol{D}$ we can find an $a > 0$ so that $aF \in N$.*

(b) *For each $\alpha > 0$ there exist positive numbers β'' and M'' so that the conditions $F \in \boldsymbol{D}_\alpha$,*

$$\max_{z \in R} |F(z)| \leqq M'', \qquad \max_{z \in R} |z_j^{\beta''} F(z)| \leqq M'' \text{ for } j = 1, 2, \cdots, n$$

imply $F \in N$.

By (a) it follows that the sets N define a locally convex topology λ on the set of functions in $\boldsymbol{D}$. By (b) together with Theorem 1 of [2] (see p. 887) it follows that for any $\alpha > 0$ the topology induced by λ on the set of functions of $\boldsymbol{D}_\alpha$ is weaker than (or the same as) the topology of the space $\boldsymbol{D}_\alpha$; trivial considerations show that this topology is the same as that of $\boldsymbol{D}_\alpha$. This proves (see [2]) that the topology λ is weaker than (or the same as) the topology of $\boldsymbol{D}$. In order to conclude the proof of Theorem 1, we must show that every neighborhood of zero in $\boldsymbol{D}$ contains some N. For this purpose, we shall need the following

LEMMA 1. *Let δ_l^k be the real n-chains defined as above. Then for any k, l, and any $F \in \boldsymbol{D}$, we have*

$$\int_{\delta_l^k} F(z)\,dz = \int_R F(z)\,dz \tag{11}$$

where the integral on the left side of (11) *converges absolutely.* (*We also set* $\delta_0^k = R$ *for all* k.) *Moreover, for each* l *we can find a positive number* θ_l *which is independent of* a_l *so that for any continuous function* H *on* C, *the conditions*

$$\max_{z \in \delta_l^k} |z_j^s H(z)| \leqq \epsilon_l \tag{12}$$

for any k, *and for* $j = 1, 2, \cdots, n$, $s = 0, 1, 2, \cdots, 2n$, *imply*

$$\int_{\delta_l^k} |H(z)|\, dz \leqq 1. \tag{13}$$

Let us assume Lemma 1 for the present, and complete the proof of Theorem 1. Let Q be any neighborhood of zero in $\boldsymbol{D}$, and denote by $\tilde{Q}$ the set of inverse Fourier transforms of the functions of Q. By well-known properties of the topology of $\mathcal{D}$, we may assume that $\tilde{Q}$ is given as follows: For each integer $r \geqq 0$, we are given a positive integer u_r and a positive number v_r, where we may assume $u_r \geqq u_{r-1}$ and $v_r \leqq v_{r-1}$ for all r. $\tilde{Q}$ consists of all $f \in \mathcal{D}$ which satisfy

$$\max_{x \notin K_{12r(n+1)}} |((\partial^s/\partial x_j^s) f)(x)| \leqq v_r \tag{14}$$

for all r and for $j = 1, 2, \cdots, n$, $s = 0, 1, 2, \cdots, u_r$, and also

$$\max_{z \in R} |((\partial^s/\partial x_j^s) f)(x)| \leqq v_0 \tag{15}$$

for $j = 1, 2, \cdots, n$ and $s = 0, 1, 2, \cdots, u_0$.

We define the set N as follows: Let $c = \eta = 1$; set $d_l = l$; let $b_l = 2n + u_l$; next choose $a_l \geqq 1$ so that $e^{-a_l} \leqq v_l \theta_l$ and so that $\{a_l\}$ is monotonically increasing to infinity.

We claim that $N \subset Q$. First, it is clear that for any $G \in N$, the inverse Fourier transform g of G satisfies (15). Let $x \in R$, $x \notin K_{12l(n+1)}$. Let k be chosen so that $\operatorname{sgn} p_{kj} = \operatorname{sgn} x_j$ for $j = 1, 2, \cdots, n$. By Lemma 1 we may write, for $s' = 0, 1, 2, \cdots, u_l$, $j = 1, 2, \cdots, n$,

$$\begin{aligned}(\partial^{s'}/\partial x_j^{s'} g)(x) &= \int_{\delta_l^k} i^{s'} z_j^{s'} G(z) \exp(ix \cdot z)\, dz \\ &= \int_{\delta_l^k} i^{s'} z_j^{s'} G(z) \exp(\tfrac{1}{2}(ix \cdot z)) \exp(\tfrac{1}{2}(ix \cdot z))\, dz.\end{aligned} \tag{16}$$

Since $x \notin K_{12l(n+1)}$ (we assume $l \geqq 1$), for at least one j, we have $|x_j| > 12(n+1)l$. Thus, for $z \in \delta_l^k$, $\mathfrak{I}(\frac{1}{2}x \cdot z) \geqq 6(n+1)la_l \geqq a_l$. This means that, for $z \in \delta_l^k$, we have

$$|\exp(\tfrac{1}{2}(ix \cdot z))| \leqq e^{-a_l} \leqq \theta_l v_l. \tag{17}$$

On the other hand, for $z \in \delta_l^k$, if $s'' = 0, 1, \cdots, u_l + 2n$, $j = 1, 2, \cdots, n$, we have

$$|z_j^{s''} G(z) \exp(\tfrac{1}{2}(ix \cdot z))|$$
$$= |G(z) \exp(3ilp_k \cdot z) \exp(\tfrac{1}{2}(ix) \cdot z - 3ilp_k \cdot z) z_j^{s''}|.$$

For $z \in \delta_l^k$, $s'' = 0, 1, 2, \cdots, u_l + 2n$, $j = 1, 2, \cdots, n$, we have

$$|G(z) \exp(3ilp_k \cdot z) z_j^{s''}| \leqq 1,$$

by the definition of N. On the other hand, for $x \notin K_{12l(n+1)}$, for some j, $|x_j| \geqq 12l(n+1)$, so that, for $z \in \delta_l^k$,

$$|\mathfrak{J}(\tfrac{1}{2}x \cdot z - 3lp_k \cdot z)| \geqq 6l(n+1)|\mathfrak{J}(z_j)| - 3ln\,|\mathfrak{J}(z_j)| > 0$$

because, for any point z on δ_l^k, and any j', j'', $|\mathfrak{J}(z_{j'})| = |\mathfrak{J}(z_{j''})|$. Thus,

$$|z_j^{s''} G(z) \exp(\tfrac{1}{2}ix \cdot z)| \leqq 1. \tag{18}$$

Now, using (16), (17), (18) and Lemma 1, we deduce that, for $s' = 0, 1, 2, \cdots, u_l$, $j = 1, 2, \cdots, n$, $x \notin K_{12l(n+1)}$

$$|((\partial^{s'}/\partial x_j^{s'}) g)(x)| \leqq v_l. \tag{19}$$

This means that $g \in \bar{Q}$ so that $G \in Q$. This completes the proof that $N \subset Q$.

Proof of Lemma 1. We assume $k = 1$, and we write δ_l for δ_l^1; the case $k \neq 1$ is handled similarly. δ_l is (for z large) the set of $z \in C^1$ such that $\mathfrak{J}(z_j) = a_l + \gamma_l$, where $\gamma_l =$ constant $\log |\zeta|$, and where $\zeta = (R(z_1), \cdots, R(z_n))$. It follows easily that the element of surface area on δ_l is $\leqq$ constant $d\zeta$ where $d\zeta$ denotes the element of area on R (for $|\zeta|$ large). The existence of θ_l now results immediately.

In order to complete the proof of Lemma 1, we must show the following: For any $\psi > 0$, call V_ψ the real n-chain in C^1 consisting of all $z \in C^1$ such that $|\zeta| = \psi$, and $0 \leqq I(z_1) = I(z_2) = \cdots = I(z_n) \leqq \gamma_l$ for all j; we want to prove that, for any $F \in \boldsymbol{D}$, $\int_{V_\psi} F(z)\,dz \to 0$ as $\psi \to \infty$. Suppose that $F \in \boldsymbol{D}_a$, and that $\gamma_l = a_l + \rho \log |\zeta|$ for $|\zeta|$ large, where ρ is some constant. Then we know that $\int_{V_\psi} dz = \rho' \psi^{n-1}(\log \psi + a_l)$ for ψ large, where ρ' is another constant.

We also know that we can find an $M > 0$ so that, for all $x \in R$, $j = 1, 2, \cdots, n$, we have $|x_j^{3n+[\alpha\rho]+2} F(x)| \leqq M$ where $[\alpha\rho]$ is the greatest integer

$\leqq \alpha_\rho$. Thus, by the Phragmén-Lindelöf theorem, for all $z \in V_\psi$, we have, for $j = 1, 2, \cdots, n$,

$$|x_j^{3n+[a\rho]+2} F(z)| \leqq M \exp(\alpha\gamma_l) \leqq M' |\psi|^{a\rho}$$

for some M'. It follows immediately that $\int_{V_\psi} |F(z)|\, dz \to 0$ which completes the proof of Lemma 1, and hence completes the proof of Theorem 1.

We have thus completed step (a) for the space $\mathcal{D}'$ as explained in the introduction. We shall now pass to step (b). Since the functions of $\boldsymbol{D}$ are defined by both growth conditions in all of C and additional conditions on R, we should not expect that every distribution W which satisfies $W * \mathcal{D} \subset \mathcal{D}$ should also satisfy $W * \mathcal{D}' = \mathcal{D}'$. Indeed, if $W \in \mathcal{D}$, then $W * \mathcal{D}' \subset \mathcal{E}$, (see [12]) so that certainly, $W * \mathcal{D}' \neq \mathcal{D}'$. In this case it is not difficult to see that $W * \mathcal{E} \neq \mathcal{E}$ (see [11]). From this it follows that $W * W * \mathcal{D}' \subset W * \mathcal{E} \neq \mathcal{E}$. (We have not been able to prove that $W * \mathcal{D}' \neq \mathcal{E}$, although this seems to be certainly true.) On the other hand, if D is a punctual distribution (see [3]), i.e. D is a partial differential difference operator, then we can prove that $D * \mathcal{D}' = \mathcal{D}'$.

THEOREM 2. *Let P be an exponential polynomial, $P \neq 0$. Then $PG \to G$ is a continuous linear map of $PD \to D$.*

(PD is the space of all PG, $G \in \boldsymbol{D}$ with the topology induced by $\boldsymbol{D}$. The properties of exponential polynomials which we shall use are derived in [3].)

Proof. We shall use induction on the number q of letters which occur in P. For $q = 0$, P is a non-zero constant and the result is obvious. Suppose $q > 0$ and that the theorem is proven for all exponential polynomials in fewer than q letters. Without loss in generality, we may assume that we may write

$$\begin{aligned} P(z) = {} & [P_0(z_2, \cdots, z_p) z_1^m \\ & + P_1(z_2, \cdots, z_p) z_1^{m-1} + \cdots + P_m(z_2, \cdots, z_p)] \exp(ia_1 z_1) \\ & + Q_2(z_1, \cdots, z_p) \exp(ia_2 z_1) + \cdots + Q_r(z_1, \cdots, z_p) \exp(ia_k z_1) \end{aligned}$$

where the P_j and the Q_j are exponential polynomials (the exponentials which occur in the Q_j being independent of z_1) with $P_0 \neq 0$, and where $a_1 < a_2 < \cdots < a_k$. The proof of Theorem 2 will be completed if we can show that, for some positive integer d, $PG \to P_0^d G$ is a continuous linear map of $P\boldsymbol{D}$ into $P_0^d \boldsymbol{D}$.

Let $L = a_k - a_1$. Let $z_2, \cdots, z_p$ be fixed and denote by P' the exponential polynomial in one variable: $z' \to P(z', z_2, \cdots, z_p)$. Let I be any interval of length $2(L+1)$ in the complex plane which is parallel to the real axis. By Lemma 1 of [3] we can find a point $z_0 \in I$ and constant K depending only on P such that $|P'(z_0)| \geqq K \exp(-|a_1 I(z_0)|)|P_0'|$, where $P_0' = P_0(z_2, \cdots, z_p)$. By Lemma 2 of [3] and the proof of Theorem 1 of [3] (see p. 288), we can describe about z_0 a circle of radius r' with $2(L+1) \leqq r' \leqq 4(L+1)$ such that, for all z' on this circle,

$$|P'(z')| \geqq K'(1+|z'|)^B |P_0'|^{d'} \exp(-d'|a_1 I(z')|) \tag{20}$$

where B, d' and K' are constants which depend only on P, and d' is a positive integer. We set $d = d'$.

Let N be a neighborhood of zero in $\boldsymbol{D}$; let $m = 3 + d'|a_1|$. By Theorem 1 we may assume that N is defined by sequences $\{a_l\}$, $\{b_l\}$, $\{d_l\}$ and positive numbers c, η such that N consists of all $G \in \mathbf{D}$ which satisfy, for each l,

$$\max |\exp(mid_l p_k \cdot z) z_j^s G(z)| \leqq \eta, \text{ where } z \in \Gamma_l^k, \tag{21}$$

for $k = 1, 2, \cdots, 2^n$, $j = 1, 2, \cdots, n$, and for $s = 0, 1, 2, \cdots, b_l$. (Theorem 1 is actually applicable only in case that m is replaced by 3, but it is clear from the proof of Theorem 1 that any number $\geqq 3$ will do.) We may also assume that $c \geqq 8(L+1)$.

For each l, k, let Θ_l^k be the region consisting of all $z \in C$ of the form

$$z_j = z_j' + i\xi_j, \qquad z' \in \Gamma_l^k, \qquad |\xi_j| \leqq 8(L+1). \tag{22}$$

We consider the set N' of all $G \in \boldsymbol{D}$ which satisfy, for each l,

$$\max |\exp(3id_l p_k \cdot z) z_j^s G(z)| \leqq \eta', \text{ where } z \in \Theta_l^k, \tag{23}$$

for $k = 1, 2, \cdots, 2^n$, $j = 1, 2, \cdots, n$, $s = 0, 1, 2, \cdots, b_l + [-B] + 1$, where $\eta' = \min(\eta/n, k'\eta/2n)(1 + 2^{-B})$ and where K', B are as in inequality (20). (If $B > 0$, then we can replace $[-B] + 1$ by 0.) Exactly as in the proof of Theorem 1 we can show that N' is a neighborhood of zero in $\boldsymbol{D}$.

We claim that the conditions $G \in \boldsymbol{D}$, $PG \in N'$ imply $P_0^d G \in N$. Let k, l be chosen; let z be any point in Γ_l^k. Let I be the interval in the complex plane, center z_1, length $2(L+1)$. We can find a point $z_0 \in I$ and a number r' with $2(L+1) \leqq r' \leqq 4(L+1)$ such that every point z' on the circle O, center z_0, radius r' satisfies (20). It is obvious from the definitions that, for any $z' \in O$, $(z', z_2, z_3, \cdots, z_n)$ lies in Θ_l^k. Thus, for any $z' \in O$, if $s = 0, 1, 2, \cdots, b_l$, $j = 1, 2, \cdots, n$, we have by (20),

$$(24)\quad \begin{aligned} &|P_0'|^d |z_j'^s G(z', z_2, \cdots, z_n) \exp(mid_1 p_k \cdot (z', z_2, z_3, \cdots, z_n))| \\ &\quad \leqq (1/K')|\exp(3id_1 p_k \cdot (z', z_2, \cdots, z_n)) \cdot z_j'^s| \\ &\quad \times |P(z', z_2, \cdots, z_n)(1+|z'|)^{-B} G(z', z_2, \cdots, z_n)| \leqq \eta \end{aligned}$$

because $PG \in N'$, where we have written $z_j' = z_j$ for $j > 1$, and $z_1' = z'$. Since the point z_1 is contained in O we also have, by the maximum modulus theorem,

$$(25)\quad |P_0'|^d |z_j^s G(z_1, z_2, \cdots, z_n) \exp(mid_1 p_k \cdot z)| \leqq \eta.$$

This proves that $P_0^d G \in N$.

We have shown that $PG \to P_0^d G$ is a continuous map of $P\boldsymbol{D} \to P_0^d \boldsymbol{D}$ at zero and hence, by linearity, everywhere. Since P_0^d is obviously an exponential polynomial in fewer than q letters, it follows from our induction assumption that $PG \to G$ is a continuous linear map of $P\boldsymbol{D}$ into $\boldsymbol{D}$ which is the desired result.

By means of Fourier transform we deduce immediately

COROLLARY. *For any partial differential-difference operator $D \neq 0$, $Dg \to g$ is a continuous linear map of $D\mathcal{D}$ into $\mathcal{D}$.*

THEOREM 3. *With D as above, for any $T \in \mathcal{D}'$ we can find an $S \in \mathcal{D}'$ such that $DS = T$.*

Proof. $Df \to T \cdot f$ is a continuous linear function $\tilde{S}$ on $D\boldsymbol{D}$ because it is the composition of the two continuous maps $Df \to f$ and $f \to T \cdot f$. By the Hahn-Banach theorem, $\tilde{S}$ can be extended to a continuous linear function on $\mathcal{D}$, that is, $\tilde{S}$ can be extended to an $S \in \mathcal{D}'$. For any $f \in \mathcal{D}$ we have

$$DS \cdot f = S \cdot Df = \tilde{S} \cdot Df = T \cdot f.$$

This shows that $DS = T$ which is the desired result.

The question naturally arises as to what is the class of $W \in \mathcal{E}'$ which have the property that $W * \mathcal{D}' = \mathcal{D}'$. Theorem 3 shows that any punctual distribution has this property. By reasoning in a slightly more complicated manner than that used in the proof of Theorem 2, we can deduce

THEOREM 2'. *Let W be any distribution in $\mathcal{E}'$ of the form $W = D + h$ where D is a non-zero punctual distribution and h is a function of compact carrier which is sufficiently often differentiable.*[3] *Then $W * g \to g$ is a continuous linear map of $W * \mathcal{D} \to \mathcal{D}$.*

THEOREM 3'. *With W as above, given any $T \in \mathcal{D}'$ we can find an $S \in \mathcal{D}'$ which satisfies $W * S = T$.*

[3] We have not determined the best possible result.

3. Division in the space $\mathcal{E}$. We showed in a previous paper (see [13]) that every partial differential-difference operator maps $\mathcal{E}$ onto $\mathcal{E}$. We want to give a proof of this fact which is in keeping with the outline proposed in the introduction. For this purpose we must compute explicity the topology of the space $\boldsymbol{E}'$ (see [5]) which is the Fourier transform of the space $\mathcal{E}'$ of Schwartz; the characterization of this topology is itself of interest. (We write $|z| = \max(|z_j|.)$

THEOREM 4. *Let H be any continuous positive function on C such that, for any $k = 1, 2, \cdots, 2^n$ and for any $t > 0$*

$$(1 + |z|^t) \exp(-itp_p \cdot z) = O(H(z)). \tag{26}$$

Let N be the set of $F \in \boldsymbol{E}'$ such that, for any $z \in C$,

$$|F(z)| \leqq H(z). \tag{27}$$

Then the sets N form a fundamental system of neighborhoods of zero in $\boldsymbol{E}'$.

Proof. Let B be a bounded set in $\boldsymbol{E}'$. It is proven in [5] that we can find a positive number t' such that every $F \in B$ satisfies

$$|F(z)| \leqq t'(1 + |z|^{t'}) \exp(-it'p_k \cdot z)$$

for $z \in C^k$, $k = 1, 2, \cdots, 2^n$. Thus, given any set N as described in the statement of Theorem 4, we can find a positive number a such that $aB \subset N$, that is, N swallows every bounded set in $\boldsymbol{E}'$ (see [5]). Since the space $\boldsymbol{E}'$ is bornologic (see [7]) it follows that N is a neighborhood of zero in $\boldsymbol{E}'$.

Conversely, let Q be any neighborhood of zero in $\boldsymbol{E}'$. Now, (see [5]) the topology of $\boldsymbol{E}'$ can be described as follows: We consider each element of $\boldsymbol{E}'$ as defining, by multiplication, a continuous linear map of $\boldsymbol{D}_F$ into $\boldsymbol{D}_F$; then the topology of $\boldsymbol{E}'$ is that obtained by giving this set of maps the compact-open topology. We want to produce a neighborhood of zero N in $\boldsymbol{E}'$ of the type described in the statement of Theorem 4 such that $N \subset Q$. By the above, it is sufficient to show that, if B is any bounded set in $\boldsymbol{D}$, and if M is a neighborhood of zero in $\boldsymbol{D}_F$, we can find a neighborhood of zero N in $\boldsymbol{E}'$, as above, such that the conditions $F \in B$, $G \in N$ imply $FG \in M$.

Now, M may be described as follows (see [7]; the result may also be deduced without much difficulty from Theorem 7 of [2], p. 893): Let H_1 be a continuous function on R with the property: For each $t > 0$, $\exp(t|x|) = O(H_1(x))$; let m be a given integer. Then M contains in the set of all $F \in \boldsymbol{D}_F$ for which

(28) $\max |z_j^s F(z)| \leqq H_1(y)$, where

$$\Im(z_1) = y_1, \Im(z_2) = y_2, \cdots, \Im(z_n) = y_n,$$

with $y = (y_1, y_2, \cdots, y_n)$, for $s = 0, 1, \cdots, m$ and for $j = 1, 2, \cdots, n$.

B may be described as follows (the proof is essentially the same as the proof of Theorem 1 of [2], p. 887): For each integer s we are given a positive number $A_s \geqq s!$; we are also given a positive number A. B is contained in the set of all $F \in \boldsymbol{D}_F$ which are of exponential type $\leqq A$ and satisfy, for all s,

(29) $$\max_{x \in R} |z_j^s F(x)| \leqq A_s; \qquad j = 1, 2, \cdots, n.$$

Now, any $F \in B$ satisfies (29) and so also satisfies, by the Phragmén-Lindelöf theorem,

(30) $$\max_{z \in C^k} |z_j^s F(z) \exp(iAp_k \cdot z)| \leqq A_s$$

for $j = 1, 2, \cdots, n$, $k = 1, 2, \cdots, 2^n$, and all s.

Define the function H_2 on R by

(31) $$H_2(x) = \sum_{r=0}^{\infty} \sum_{j=1}^{n} |x_j|^r / 2^{r+1} n A_r.$$

Since $A_r \geqq r!$, this series converges for all r. Then (31) and (29) imply that, for any $x \in R$, $F \in B$, we have $|F(x)H_2(x)| \leqq 1$. We set

(32) $$H(z) = H_2(x)H_1(y)|\exp(iAp_k \cdot z)|/(1 + |z|^m)$$

for any $z \in C^k$, $k = 1, 2, \cdots, 2^n$, where

$$x = (\Re(z_1), \cdots, \Re(z_n)), \qquad y = (\Im(z_1), \cdots, \Im(z_n)).$$

It is clear from (30), (31), and (32) that, for any $z \in C$, for any $j = 1, 2, \cdots n$, and for any $F \in B$, we have

(33) $$|z_j^s H(z)F(z)| \leqq H_1(y)$$

for $s = 0, 1, \cdots, m$. Using (33) we deduce that the set N of $G \in \boldsymbol{D}_F$ which satisfy $|G(z)| \leqq H(z)$ for all $z \in C$ has the property that $NB \in M$. Since H clearly satisfies the relation (26), this completes the proof of Theorem 4.

THEOREM 5. *Let $W \in \mathcal{E}'$ be of the form $W = D + f$ where D is a non-zero partial differential-difference operator with constant coefficients (punctual distribution) and $f \in \mathcal{E}'$is a sufficiently often differentiable function.*[4] *Then $W * U \to U$ is a continuous linear map of $W * \mathcal{E}'$ into $\mathcal{E}'$.*

[4] The same remarks apply as in footnote 2.

The proof of Theorem 5 proceeds along essentially the same lines as the proof of Theorem 2 (and is, in fact, somewhat easier); we shall omit the details.

By the methods of proof of Theorem 3 we can now deduce (see [3])

THEOREM 6. *With W as above, for any $f \in E$ there is a $g \in E$ such that $W * g = f$.*

4. Division in the space H. By $\mathcal{H}$ we denote the space of entire functions on C with its usual topology (see [4]). Each element of $\mathcal{H}'$ can be represented as a distribution of compact carrier on C. The Fourier transform H' of $\mathcal{H}'$ is the space of all entire functions of exponential type on C with no extra conditions on R. As mentioned in the introduction, we should therefore expect that every $W \in \mathcal{H}'$ has the property that $W * U \to U$ is a continuous linear map of $W * \mathcal{H}'$ into $\mathcal{H}'$. That this is actually the case will be seen later (see the corollary to Theorem 8 below). First we shall have to describe the topology of H' explictly:

THEOREM 7. *Let H be any continuous positive function on C such that, for any $t > 0$,*

$$\exp(t|z|) = O(H(z)). \tag{34}$$

We consider the set N of $F \in H'$ such that $|F(z)| \leqq H(z)$. Then these sets N form a fundamental system of neighborhoods of zero in H'.

Proof. First, let N be a set satisfying the hypotheses of Theorem 7. Then N is clearly convex. If B is any bounded set in H' then (the proof is similar to that of Proposition 2 of [4], p. 296) we can find positive numbers b, c so that all $F \in B$ satisfy $|F(z)| \leqq b \exp(c|z|)$. Thus, by (34), we can find an $a > 0$ such that $aB \subset N$, that is, N swallows every bounded set in H' (see [5]). Now, H' is bornologic (see [9]); this means that N is a neighborhood of zero in H'.

(We can also prove that N is a neighborhood of zero in H' by using the method of proof of Theorem 14 below, without using the results of Grothendieck. This method is of use when we do not know in advance that the space in question is bornologic. Actually, we could deduce in this manner that H' is bornologic.)

Conversely, let M be a neighborhood of zero in H'; we want to produce a neighborhood of zero N of the type described in the statement of Theorem 7 such that $N \subset M$. Call Q the set of inverse Fourier transforms of the functions of M (see [4]). By the definition of the topology of a dual space, we

can find a bounded set K in $\mathcal{H}$ so that the conditions $U \in \mathcal{H}'$, $|U \cdot f| \leqq 1$ for all $f \in K$ imply $U \in Q$.

Now (see [4], Proposition 1 and Equation (1) on p. 296), if we write $G \in \boldsymbol{H}'$ in the form

$$G(z) = \sum (i^{k_1+k_2+\cdots+k_n}/k_1!k_2!\cdots k_n!)\ G_{k_1k_2\cdots k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$$

and if we write $f \in \mathcal{H}$ in the form $f(z) = \sum f_{k_1k_2\cdots k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$ then we have

$$F^{-1}G \cdot f = \sum G_{k_1k_2\cdots k_n} f_{k_1k_2\cdots k_n}. \tag{35}$$

Since K is bounded in $\mathcal{H}$, for each system of positive integers l we can find a positive number $c_{l_1l_2\cdots l_n}$ such that every $f \in K$ satisfies

$$\max |f(z)| \leqq c_{l_1l_2\cdots l_n}, \text{ where } |z_1| \leqq l_1, |z_2| \leqq l_2, \cdots, |z_n| \leqq l_n. \tag{36}$$

From (36) and Cauchy's formula we deduce that the inequality

$$|f_{k_1k_2\cdots k_n}| \leqq c_{l_1l_2\cdots l_n}/l_1^{k_1}l_2^{k_2}\cdots l_n^{k_n} \tag{37}$$

holds for all $f \in K$ and for all integers $k_1, k_2, \cdots, k_n$, $l_1, l_2, \cdots, l_n$. We may clearly assume that $c_{11\cdots1} = 1$ and that $c_{l_1l_2\cdots l_n} \leqq c_{l_1'l_2'\cdots l_n'}$ if

$$l_1 \leqq l_1', l_2 \leqq l_2', \cdots, l_n \leqq l_n'.$$

Let $\{\gamma_j\}$ be a strictly increasing sequence of numbers with $\gamma_0 = 0$. Let $\{d_j\}$ be a sequence of non-negative numbers with $d_0 = 0$, $d_j > 0$ for $j > 0$. We require further that

(a) $$d_j + \gamma_j \leqq \gamma_{j+1}$$

for any j. We call $\alpha_{j_1j_2\cdots j_n}$ the region consisting of all $y \in C$ for which $d_{j_k} + \gamma_{j_k} \leqq |y_k| \leqq \gamma_{j_k+1}$.

(b) $$\exp[\tfrac{1}{2}(j_1|y_1| + j_2|y_2| + \cdots + j_n|y_n|)]/c_{j_1+1,\cdots,j_n+1} \geqq \exp[\tfrac{1}{2}((j_1-1)|y_1| + (j_2-1)|y_2| + \cdots + (j_n-1)|y_n|)],$$

whenever $j_1 \geqq 1, j_2 \geqq 1, \cdots, j_n \geqq 1$, and whenever $|y_k| \geqq d_{j_k} + \gamma_{j_k}$ for all k.

(c) There is no point of the form l/m in the interval $\gamma_j \leqq x \leqq \gamma_j + d_j$ for any j, where m, l are integers and $m \leqq 2(j+1)$.

The existence of the sequences $\{\gamma_j\}$, $\{d_j\}$ is obvious. Moreover, it is clear that the regions $\alpha_{j_1,j_2,\cdots,j_n}$ do not overlap.

We set

$$c_{j_1+1\ j_2+1\ \cdots\ j_n+1}\ H(z) = \exp(\tfrac{1}{2}(j_1|z_1| + j_2|z_2| + \cdots + j_n|z_n|)), \quad z \in \alpha_{j_1j_2\cdots j_n} \tag{38}$$

and H is defined so as to be continuous and monotonic (i.e. $H(z) \leqq H(z')$ if $|z_j| \leqq |z_j'|$ for $j = 1, 2, \cdots, n$). It is obvious from (38) and (b) above that H satisfies (34).

Denote by N the set of all $G \in \boldsymbol{H}'$ for which $|G(z)| \leqq H(z)$ for all $z \in C$. Let $G \in N$ and let $(k_1, k_2, \cdots, k_n)$ be an n-tuple of non-negative integers. It is not difficult to see by (c) that we can find an n-tuple of integers $(l_1, l_2, \cdots, l_n)$ and an n-tuple of positive numbers $(a_1, a_2, \cdots, a_n)$ such that

$$(39) \quad k_1/l_1 \geqq a_1 \geqq k_1/l_1 + 1,$$
$$k_2/l_2 \geqq a_2 \geqq k_2/l_2 + 1, \cdots, k_n/l_n \geqq a_n \geqq k_n/l_n + 1$$

and such that the n-chain Ξ in C defined by $|z_1| = a_1$, $|z_2| = a_2, \cdots, |z_n| = a_n$ lies in $\alpha_{l_1 l_2 \cdots l_n}$.

By Cauchy's formula we may write [5]

$$G_{k_1 k_2 \cdots k_n} = -ik_1!k_2! \cdots k_n! \int_{\Xi} G(z)/z_1^{k_1+1} z_2^{k_2+1} \cdots z_n^{k_n+1}\, dz.$$

Now, on Ξ, we have

$$|G(z)| \leqq H(z) = \exp\left(\tfrac{1}{2}(a_1 l_1 + a_2 l_2 + \cdots + a_n l_n)\right)/c_{l_1+1, l_2+1, \cdots, l_n+1}$$

by (38). Thus, on making use of (39) we find

$$c_{l_1+1, l_2+1, \cdots, l_n+1} k_1^{k_1} k_2^{k_2} \cdots k_n^{k_n} |G_{k_1 k_2 \cdots k_n}|$$
$$\leqq k_1!k_2! \cdots k_n! \exp\left(\tfrac{1}{2}(k_1 + k_2 + \cdots + k_n)\right)(l_1+1)^{k_1}(l_2+1)^{k_2} \cdots (l_n+1)^{k_n}.$$

By Stirling's formula, we can find a constant θ such that

$$(40) \quad c_{l_1+1, l_2+1, \cdots, l_n+1} |G_{k_1 k_2 \cdots k_n}|$$
$$\leqq \theta \exp\left(-\tfrac{1}{2}(k_1 + k_2 + \cdots + k_n)\right)(l_1+1)^{k_1}(l_2+1)^{k_2} \cdots (l_n+1)^{k_n}.$$

Using (35), (37), and (40) it is obvious that we can find a $\theta' > 0$ such that every $G \in N$ satisfies

$$(41) \qquad F^{-1}G \cdot f \leqq \theta' \text{ for all } f \in K.$$

This proves that $(1/\theta')N \subset M$ which completes the proof of Theorem 7.

THEOREM 8. *Let $J \in \boldsymbol{H}'$, $J \neq 0$; then $JG \to G$ is a continuous linear map of $J\boldsymbol{H}'$ into $\boldsymbol{H}'$.*

[5] We normalize the measure in such a manner that the usual factor $(2\pi)^n$ does not appear.

Proof. The idea of the proof is to use the following minimum modulus theorem for entire functions of exponential type (see [4], Theorem 5, p. 317): Suppose k is an entire function of exponential type of one complex variable and that $|k(z)| \leqq M \exp(A|z|)$ for all z. Then for any $r > 0$ there is an r' with $r \leqq r' \leqq 2r$ such that

$$\min_{|z|=r'} |k(z)| \geqq |k(0)|^d \exp(B \log M + cAr') \tag{42}$$

where B and c are certain constants and d is a positive integer.

We shall prove Theorem 8 by induction on the number q of variables on which J depends. For $q = 0$, J is a non-zero constant and the result is obvious; we assume $q > 0$ and that the result is known for entire functions of exponential type in fewer than q variables. We may clearly assume that J depends on $Z_1, Z_2, \cdots, Z_q$. For any complex numbers $z_2, \cdots, z_q$, set $J_1(z_2, \cdots, z_q) = J(0, z_1, \cdots, z_q)$. It is sufficient to show that $JG \to J_1{}^d G$ is a continuous linear map of $J\boldsymbol{H}' \to J_1{}^d \boldsymbol{H}'$.

By linearity, we must show the following: Given any neighborhood of zero N in $\boldsymbol{H}'$ there exists a neighborhood of zero N_1 in $\boldsymbol{H}'$ such that the conditions $G \in \boldsymbol{H}'$, $JG \in N_1$ imply $J_1{}^d G \in N$. By Theorem 6 we may assume that N consists of all $F \in \boldsymbol{H}'$ for which $|F(z)| \leqq H(z)$ for all $z \in C$, where H is a function satisfying (34). We may clearly assume that H is monotonic, i.e.

$$H(z') \geqq H(z) \text{ if } |z_1'| \geqq |z_1|, |z_2'| \geqq |z_2|, \cdots, |z_n'| \geqq |z_n|. \tag{43}$$

Now, we can clearly find positive numbers A, M such that $|J(z)| \leqq M \exp(A|z|)$ for all $z \in C$. We define the function H_1 on C by

$$H_1(2z) = M^B \exp(2cA|z_1| + BA|z_2| + \cdots + BA|z_q|) H(z_1, z_2, \cdots, z_n) \tag{44}$$

where B and c are the constants that appear in (42). It is clear that H_1 satisfies (34). Denote by N_1 the neighborhood of zero in $\boldsymbol{H}'$ consisting of all $G \in \boldsymbol{H}'$ for which $|G(z)| \leqq H_1(z)$ for all $z \in C$. We claim that the conditions $JG \in N_1$ implies $J_1{}^d G \in N$.

For, given any $z \in C$, we can find by (42) a number r' with $|z_1| \leqq r' \leqq 2|z_1|$ and

$$\min_{|z_1'|=r'} J(z_1', z_2, \cdots, z_q) \geqq |J_1(z_2, \cdots, z_q)|^d M^B \exp(cAr' + BA|z_2| + \cdots + BA|z_q|).$$

Therefore, if $|z_1'| = r'$, and $JG \in N_1$, we have

$$|J_1(z_2, \cdots, z_q)|^d |G(z_1', z_2, \cdots, z_n)|$$
$$\leqq M^{-B} \exp(-cAr' - BA|z_2| - \cdots - BA|z_q|) \times |J(z_1', z_2, \cdots, z_q) G((z_1', z_2, \cdots, z_n)|$$
$$\leqq M^{-B} \exp(-cAr' - BA|z_2| - \cdots - BA|z_q|) H_1(z_1', z_2, \cdots, z_n).$$

Since $|z_1| \leqq r' \leqq 2|z_1|$ we have, by the maximum modulus theorem,

$$|J(z_2, \cdots, z_q)|^d |G(z_1, z_2, \cdots, z_n)|$$
$$\leqq M^{-B} \exp(-2cA|z_1| - BA|z_2| - \cdots - BA|z_q|) \times H_1(2z_1, 2z_2, \cdots, 2z_n) = H(z_1, z_2, \cdots, z_n)$$

because (see (43) and (44)) H_1 is monotonic, and $c < 0$. This completes the proof of Theorem 8.

By taking the Fourier transform we deduce

COROLLARY. *For any $W \in \mathcal{H}'$, $W \neq 0$, $W * U \to U$ is a continuous linear map of $W * \mathcal{H}'$ into $\mathcal{H}'$.*

THEOREM 9. *For any $W \in \mathcal{H}'$, $f \to W * f$ maps $\mathcal{H}$ onto $\mathcal{H}$ continuously.*

Theorem 9 follows from the corollary to Theorem 8 exactly in the same manner as Theorem 3 is deduced from the Corollary to Theorem 2.

5. Entire functions of finite order. Let f be an entire function, and let $A \geqq 1$. We say that f is of order $\leqq A$ if, for every $\epsilon > 0$, we have

$$f(z) = O(\exp |z|^{A+\epsilon}). \tag{45}$$

The greatest lower bound of A for which (45) holds is called the *order* of f. By $\mathcal{Q}_A$ we denote the space of all entire functions of order $\leqq A$; it is clear that $\mathcal{Q}_A$ is a ring under multiplication. We define a topology in $\mathcal{Q}_A$ by means of the semi-norms

$$v_\epsilon(f) = \sup_{z \in C} \exp(-|z|^{A+\epsilon}) |f(z)| \tag{46}$$

for any $f \in \mathcal{Q}_A$.

PROPOSITION 1. *$\mathcal{Q}_A$ is a complete metrizable locally convex topological vector space,* i.e. *$\mathcal{Q}_A$ is a Frechet space. $\mathcal{Q}_A$ is not a Banach space. $\mathcal{Q}_A$ is a Montel space, so that $\mathcal{Q}_A$ is reflexive. $\mathcal{Q}_A$ is also a Schwartz space.*[6]

[6] The theory of Schwartz spaces has been developed by Grothendieck in [9].

Proof. It is obvious that $\mathfrak{Q}_A$ is a locally convex topological vector space. Moreover, we may clearly define the topology of $\mathfrak{Q}_A$ by means of the seminorms $v_{1/m}$ for m integral; thus, $\mathfrak{Q}_A$ is metrizable. In order to show completeness, we may therefore restrict ourselves to sequences. If $\{f_j\}$ is a Cauchy sequence in $\mathfrak{Q}_A$, we know, from the fact that $\mathfrak{H}$ is a Frechet space (see [4]), that $\{f_j\}$ converges to a function $g \in \mathfrak{H}$ on the topology of $\mathfrak{H}$. It is clear that, for every $\epsilon > 0$, $g(z) = O(\exp(|z|^{A+\epsilon})$ so that $g \in \mathfrak{Q}_A$. It is readily verified that $\{f_j\}$ converges to g in the topology of $\mathfrak{Q}_A$. It is clear that no neighborhood of zero in $\mathfrak{Q}_A$ is bounded, so $\mathfrak{Q}_A$ is not a Banach space.

We shall show now that $\mathfrak{Q}_A$ is a Montel space, that is, the closed bounded sets of $\mathfrak{Q}_A$ are compact. We shall show even more: For each $\epsilon > 0$ the set N_ϵ of $f \in \mathfrak{Q}_A$ which satisfy $v_\epsilon(f) \leqq 1$ is precompact for the topology induced by $v_{\epsilon'}$ whenever $\epsilon' > \epsilon$; this implies that $\mathfrak{Q}_A$ is a Schwartz space. Let $\{f_j\}$ be a sequence in N_ϵ; we can extract a subsequence $\{f_j\}$ which converges in $\mathfrak{H}$, say $f_{j_k} \to f$ in $\mathfrak{H}$, because $\mathfrak{H}$ is a Montel space and N_ϵ is clearly bounded in $\mathfrak{H}$; it is obvious that $f \in N_\epsilon$. Since each $f_{j_k} \in N_\epsilon$ we deduce immediately that $v_{\epsilon'}(f_{j_k} - f) \to 0$. This completes the proof of Proposition 1.

PROPOSITION 2. *For any* $f \in \mathfrak{H}$, write $f(z) = \sum f_{k_1 k_2 \cdots k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$. *Then* $f \in \mathfrak{Q}_A$ *if and only if*

$$\liminf \log(1/|f_{k_1 k_2 \cdots k_n}|)/\{(k_1 \log k_1)(k_2 \log k_2) \cdots (k_n \log k_n)\} = 1/A. \tag{47}$$

In case $n = 1$ this Proposition is well-known (see [13], p. 253); the proof for $n > 1$ can be accomplished by a similar method.

By $\mathfrak{Q}_A'$ we denote the dual of $\mathfrak{Q}_A$ with the topology of uniform convergence on the compact (bounded) sets of $\mathfrak{Q}_A$. For any $U \in \mathfrak{Q}_A'$ and any integers $k_1, k_2, \cdots, k_n$ we set

$$U_{k_1 k_2 \cdots k_n} = U \cdot Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}.^7 \tag{48}$$

It is not difficult to show

PROPOSITION 3. *For any* $f \in \mathfrak{Q}_A$, $\sum f_{k_1 k_2 \cdots k_n} Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}$ *converges to* f *in the topology of* $\mathfrak{Q}_A$. *Thus, for any* $U \in \mathfrak{Q}_A'$,

$$U \cdot f = \sum U_{k_1 k_2 \cdots k_n} f_{k_1 k_2 \cdots k_n} \tag{49}$$

where the series on the right side of (49) *converges absolutely.*

For any $U \in \mathfrak{Q}_A'$, we can find an $\epsilon > 0$ so that U is bounded on the set of $f \in \mathfrak{Q}_A$ for which $v_\epsilon(f) \leqq 1$. By the Hahn-Banach theorem, U can

[7] For each j, Z_j is the function on C: $z \to z_j$.

be extended to a continuous linear function on the space $V_{A+\epsilon}$ of all continuous functions on C which are $o(\exp(|z|^{A+\epsilon}))$. Thus, we have

PROPOSITION 4. *For each $U \in \mathcal{Q}_A'$ we can find an $\epsilon > 0$ so that U can be represented by a measure which we again denote by U which falls off like* $\exp(-|z|^{A+\epsilon})$, that is,

$$\int_{|z|\leqq r} d|U(z)| = O(\exp(-r^{A+\epsilon})). \tag{50}$$

If B is a bounded set in $\mathcal{Q}_A'$, then the ϵ and symbol "O" in (50) *can be chosen uniformly for $U \in B$.*

For any $U \in \mathcal{Q}_A$, we define the *Fourier transform* of U as the function on C:

$$f \to U \cdot \exp(iz\cdot) = \int \exp(iz\cdot t)\,dU(t) = (\boldsymbol{F}(U))(z). \tag{51}$$

It follows immediately from Proposition 4 that $\boldsymbol{F}(U)$ is an entire function.

We define A' by $1/A + 1/A' = 1$, that is, A' is the conjugate exponent of A (see [15]). We wish to recall Hölder's inequality (see [13], p. 383 inequality (5)): For any complex numbers x, y we have

$$|x|^A/A + |y|^{A'}/A' \geqq |xy|. \tag{52}$$

Using the fact that for any z, $t \in C$ we have $|z\cdot t| \leqq |z|\,|t|$, we deduce from (52)

$$|z|^A/A + |t|^{A'}/A' \geqq |z\cdot t| \text{ for any } z, t \in C. \tag{52'}$$

PROPOSITION 5. *For any $U \in \mathcal{Q}_A'$, we can find an $\epsilon' > 0$ such that $\boldsymbol{F}(U)$ is an entire function of order $\leqq A' - \epsilon'$.*[8] *If B is a bounded set in $\mathcal{Q}_A'$ we can choose ϵ' uniformly for $U \in B$. We may write:*

$$(\boldsymbol{F}(U))(z) = \sum (i^{k_1+k_2+\cdots+k_n}/k_1!k_2!\cdots k_n!)(U_{k_1k_2\cdots k_n} z_1^{k_1} z_2^{k_2}\cdots z_n^{k_n}). \tag{53}$$

Proof. We apply Proposition 3 to equation (51), using the fact that

$$\exp(iz\cdot) = \sum (i^{k_1+k_2+\cdots+k_n}/k_1!k_2!\cdots k_n!)\,(z_1Z_1)^{k_1}(z_2Z_2)^{k_2}\cdots(z_nZ_n)^{k_n}$$

and we deduce (53) immediately. The first part of Proposition 5 can now be deduced from the fact that the numbers $U_{k_1k_2\cdots k_n}$ render the right side of (49) convergent whenever $f \in \mathcal{Q}_A$ if and only if, for some $\epsilon' > 0$,

$$\sum (i^{k_1+k_2+\cdots+k_n}/k_1!k_2!\cdots k_n!)\,U_{k_1k_2\cdots k_n} Z_1^{k_1} Z_2^{k_2}\cdots Z_n^{k_n}$$

---

[8] In case $A = 1$, $A' = \infty$, and $A' - \epsilon'$ is to be interpreted as some (finite) number. A similar convention is to be employed throughout the rest of this paper.

defines an entire function of order $\leqq A'-\epsilon'$. (This fact can be proved by the methods of [7].) The part of Proposition 5 referring to the bounded set B can be handled similarly.

Another proof of the first part of Proposition 5 can be obtained from Hölder's inequality (52′), and Proposition 4. Thus, if ϵ' is determined by the condition $1/(A+\epsilon)+1/(A'-\epsilon')=1$, then we have, for any $z\in C$,

$$(54)\qquad (\boldsymbol{F}(U))(z)=\int \exp(iz\cdot t)\,dU(t)\leqq\int \exp(|z\cdot t|\,d\,|U(t)|$$
$$\leqq\int \exp(|z|^{A'-\epsilon'}/(A'-\epsilon')+t^{A+\epsilon}/(A+\epsilon))d|U(t)|$$
$$=\exp(z^{A'-\epsilon'}/(A'-\epsilon'))\int \exp(t^{A+\epsilon}/(A+\epsilon))d\,|U(t)|=O(\exp(z^{A'-\epsilon'})).$$

Similarly, if U lies in a bounded set b in $\mathcal{Q}_A'$, the symbol "O" in (54) can be chosen uniformly for $U\in B$. This completes the proof of Proposition 5.

By means of Hölder's inequality (52′) we can also establish easily

Proposition 6. *For each $\epsilon>0$ there exists a constant $c_\epsilon>0$ such that, for all $z\in C$,*

$$(55)\qquad v_\epsilon(\exp(iz\cdot))\leqq c_\epsilon|z|^{A'-\epsilon'}$$

where, $1/(A+\epsilon)+(A'-\epsilon')=1$.

Remark. Proposition 6 furnishes us with another proof of the first part of Proposition 5.

We now want to derive the converse of Proposition 5, that is, $\boldsymbol{Q}_A'=\boldsymbol{F}(\mathcal{Q}_A')$ consists of all entire functions of order $<A'$. We also want to describe completely the topology of $\boldsymbol{Q}_A'$ in order to study the division problem in the space $\mathcal{Q}_A$. ($\boldsymbol{Q}_A'$ is given the topology to make $\boldsymbol{F}$ a topological isomorphism.)

Proposition 7. *Let B be a set in $\boldsymbol{Q}_A'$; then B is bounded in $\boldsymbol{Q}_A'$ if and only if, for some $\epsilon'>0$ we can find an $M>0$ such that*

$$|G(z)|\leqq M\exp(|z|^{A'-\epsilon'})\ \textit{for all}\ z\in C,\ G\in B.$$

Proof. The necessity of the condition was established in Proposition 5. Suppose then that B satisfies the stated condition; let ϵ be defined by $1/(A+\epsilon)+1/(A'-\epsilon')=1$. (We assume, as we may, that $A'-\epsilon'>1$.) We claim that, for some $M'>0$, $|\boldsymbol{F}^{-1}G\cdot f|\leqq 1$ for all $f\in N_{\epsilon/2}$. This is, in fact, an easy consequence of Proposition 3 and Cauchy's formula.

PROPOSITION 8. *Let* $\{G^j\}$ *be a sequence in* $\boldsymbol{Q}_A'$. *A necessary and sufficient condition that* $G^j \to 0$ *in* $\boldsymbol{Q}_A'$ *is that there exist an* $\epsilon' > 0$ *such that*

$$\max_{z \in C} \exp(-|z|^{A'-\epsilon'} G^j(z)) \to 0. \tag{56}$$

For each j, *denote by* U^j *the inverse Fourier transform of* G^j. *Then if* (56) *is satisfied, we even have* $U^j \cdot f \to 0$ *uniformly on the set of* $f \in N_{\epsilon/2}$.

Proof. Suppose first that $G^j \to 0$ in $\boldsymbol{Q}_A'$. Then it follows from Proposition 1 that, for some $\epsilon_1 > 0$, $U^j \cdot f \to 0$ uniformly for $f \in N_{\epsilon_1}$. (See [9]; see also [12], vol. I, p. 91, where a similar result is proven for the space $\mathcal{E}$ of Schwartz.) By use of Proposition 6 we can now deduce (54).

The converse is established by the method of proof of Proposition 7.

From Proposition 8 we deduce immediately

COROLLARY. *Let* $\{G^j\}$ *be a sequence in* $\boldsymbol{Q}_A'$ *and let* M, ϵ' *be positive numbers such that, for all* j,

$$|G^j(z)| \leqq b \exp(|z|^{A'-\epsilon'}) \text{ for any } z \in C. \tag{57}$$

Suppose also that $G^j \to 0$ *in the topology of* $\mathcal{H}$. *Then also* $G^j \to 0$ *in the topology of* $\boldsymbol{Q}_A'$.

THEOREM 10. $\boldsymbol{Q}_A'$ *consists of all entire functions of order* $< A'$.

Proof. The fact that every function in $\boldsymbol{Q}_A'$ is an entire function of order $< A'$ was established in Proposition 5, so we proceed to the converse. Denote by K the vector space of all entire functions of order $< A'$; we have $\boldsymbol{Q}_A' \subset K$, and we want to show $\boldsymbol{Q}_A' = K$. Clearly, every polynomial lies in $\boldsymbol{Q}_A'$; we shall show that, for any $G \in K$ we can find a sequence of polynomials G^j which converges to G in the topology of $\boldsymbol{Q}_A'$. Since the space $\boldsymbol{Q}_A'$ is known to be complete, this will show that $G \in \boldsymbol{Q}_A'$.

Let us write $G(z) = \sum G_{k_1 k_2 \cdots k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}$. Then we set

$$G^j = \sum_{k_1+k_2+\cdots+k_n \leqq j} G_{k_1 k_2 \cdots k_n} Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}.$$

It is clear that $G^j \to G$ in the topology of $\mathcal{H}$. Moreover, it follows from the Corollary to Proposition 8 that $\{G^j\}$ is a Cauchy sequence in $\boldsymbol{Q}_A'$; thus, we can find a $G_1 \in \boldsymbol{Q}_A'$ such that $G^j \to G_1$ in the topology of $\boldsymbol{Q}_A'$. Now, for each $z \in C$, and each j,

$$G^j(z) = \boldsymbol{F}^{-1}(G^j) \cdot \exp(iz\,\cdot) \to \boldsymbol{F}^{-1}(G_1) \cdot \exp(iz\,\cdot) = G_1(z).$$

Thus, $G_1 = G$ which completes the proof of Theorem 10.

For any $f \in \mathcal{Q}_A$, $z \in C$, we denote by $\tau_z f$ the function on C: $(\tau_z f)(t)$

$= f(t-z)$; it is clear that $\tau_z f \in \mathcal{Q}_A$. For any $f \in \mathcal{Q}_A$, $U \in \mathcal{Q}_A'$, we define the *convolution* $U * f$ by $(U * f)(z) = U \cdot \tau_{-z} f$ for any $z \in C$. According to Proposition 4 we may write

$$(58) \qquad (U * f)(z) = \int f(t+z)\, dU(t)$$

where, for some $\epsilon > 0$, the measure U falls off like $\exp(-|t|^{A+\epsilon})$. It follows easily from (58) that $U * f \in \mathcal{Q}_A$.

PROPOSITION 9. *If B is any bounded set in $\mathcal{Q}_A$, then $U \to U * f$ are, for $f \in B$, equicontinuous linear maps of $\mathcal{Q}_A' \to \mathcal{Q}_A$. If K is any bounded set in $\mathcal{Q}_A'$, then $f \to U * f$ are, for $U \in K$, equicontinuous linear maps of $\mathcal{Q}_A \to \mathcal{Q}_A$.*

Proof. By bilinearity, we need verify continuity only at zero. Let m be a fixed positive integer; since B is bounded in $\mathcal{Q}_A$, for each integer $j \geqq m$ we can find a positive number M_j such that $v_{1/j}(f) \leqq M_j$ for all $f \in B$. Call B' the set of $f \in \mathcal{Q}_A$ such that $v_{1/j}(f) \leqq M_j'$ for all $j \geqq m$, where $M_j' = M_j \exp(2^{A+1/m})$. Let N be the neighborhood of zero in $\mathcal{Q}_A'$ consisting of all $U \in \mathcal{Q}_A'$ such that $|U \cdot f| \leqq 1$ for all $f \in B'$; we claim that, for $U \in N$, $f \in B$, we have $U * f \in N_{1/m}$.

Let us note that $t \to t^{A+1/m}$ is a convex function on C. Thus, for any $t, z \in C$, any $f \in B$, and any $j > 0$, we have

$$(59) \qquad \begin{aligned} |(\tau_{-z} f)(t)| = |f(z+t)| &\leqq M_j \exp(|z+t|^{A+1/j}) \\ &\leqq M_j \exp(2^{A+1/j}(|z|^{A+1/j} + |t|^{A+1/j})). \end{aligned}$$

Thus, $\exp(-|z|^{A+1/j})(\tau_{-z} f)(t) \leqq M_j' \exp(|t|^{A+1/j})$, which implies that $\exp(-|z|^{A+1/m})\tau_{-z} f \in B'$ for all $z \in C$. Hence, for all $U \in N$, $f \in B$, $z \in C$, we have

$$|(U * f)(z)| = |U \cdot \tau_{-z} f| \leqq \exp(|z|^{A+1/m}),$$

that is, $U * f \in N_{1/m}$. This completes the proof of the first part of Proposition 9.

Let $\epsilon > 0$ be fixed. Let N' be the set of all $f \in \mathcal{Q}_A$ such that $|U \cdot f| \leqq 1$ for all $U \in K$; by the reflexivity of $\mathcal{Q}_A$ (Proposition 1), N' is a neighborhood of zero in $\mathcal{Q}_A$. Thus, we can find an $\epsilon_1 > 0$ and an $a > 0$ such that $N' \subset a N_{\epsilon_1}$; call $\epsilon_2 = \min(\epsilon, \epsilon_1)$. We claim that, for $f \in \exp(-2^{A+\epsilon_2}) a N_{\epsilon_2}$, $U \in K$, we have $U * f \in N_\epsilon$.

As in (59) above, if $z, t \in C$, $f \in \exp(-2^{A+\epsilon_2}) a N_{\epsilon_2}$,

$$|(\tau_{-z} f)(t)| \leqq a \exp(|z|^{A+\epsilon_2} + |t|^{A+\epsilon_2})$$

so that $\exp(-|z|^{A+\epsilon_2})\tau_{-z}f \in aN_{\epsilon_2} \subset N'$; *a fortiori*, $\exp(-|z|^{A+\epsilon})\tau_{-z}f \in N'$. Thus, for any $U \in K$, $f \in \exp(-2^{A+\epsilon_2})aN_{\epsilon_2}$, we have $U * f \in N_\epsilon$ which completes the proof of Proposition 9.

For any $U, V \in \mathcal{Q}_A'$, the convolution $U * V$ is defined by $U * V \cdot f = U \cdot V * f$ for any $f \in \mathcal{Q}_A$. $U * V$ is in $\mathcal{Q}_A'$ by Proposition 9.

From Proposition 9 above and the method of proof of Propositions 7 and 8 and Theorem 2 of [4] (see pp. 300-302), we can deduce

PROPOSITION 10. *$(U, V) \to U * V$ is a continuous bilinear map of $\mathcal{Q}_A' \times \mathcal{Q}_A' \to Q_A'$. Moreover, $U * V = V * U$.*

PROPOSITION 11. *For any $U \in \mathcal{Q}_A'$, $f \in \mathcal{Q}_A$, and any integers $k_1, k_2, \cdots k_n$, we have*

$$(U * f)_{k_1 k_2 \cdots k_n} = \sum U_{l_1 l_2 \cdots l_n} f_{l_1+k_1, l_2+k_2, \cdots, l_n+k_n} \tag{60}$$

where the sum on the right is extended over all integers $l_1, l_2, \cdots, l_n$ and where the series on the right converges absolutely.

PROPOSITION 12. *For any $U, V \in \mathcal{Q}_A'$,*

$$\boldsymbol{F}(U * V) = \boldsymbol{F}(U)\boldsymbol{F}(V). \tag{61}$$

COROLLARY. *For $U, V, W \in \mathcal{Q}_A'$, $U * (V * W) = (U * V) * W$, that is, convolution is associative.*

We are now in a position to study the division problem in the space $\mathcal{Q}_A$. Since $\boldsymbol{Q}_A'$ is defined by growth conditions in C only (with no additional conditions on R) we should expect that for every $W \in \mathcal{Q}_A'$, $W * \mathcal{Q}_A = \mathcal{Q}_A$. That this is actually the case will be seen later (Theorem 13 below). First, we shall characterize completely the topology of the space $\boldsymbol{Q}_A'$.

THEOREM 11. *Let H be a continuous function on C such that, for every $\epsilon' < 0$,*

$$\exp(|z|^{A'-\epsilon'}) = O(H(z)). \tag{62}$$

Call N the set of $G \in \boldsymbol{Q}_A'$ such that $|G(z)| \leqq H(z)$ for all $z \in C$. Then these sets N form a fundamental system of neighborhoods of zero in $\boldsymbol{Q}_A'$.

Proof. Let N be a set satisfying the above hypotheses. By Proposition 7 and (62), if B is any bounded set in $\boldsymbol{Q}_A'$ we can find a positive number b such that $bB \subset N$, that is, N swallows every bounded set in $\boldsymbol{Q}_A'$. It follows from Proposition 1 and the results of [9] that $\boldsymbol{Q}_A'$ is bornologic. Thus, since N is clearly convex, N is a neighborhood of zero in $\boldsymbol{Q}_A'$.

A direct proof of the fact that N is a neighborhood of zero in $\boldsymbol{Q}_A'$ (without using the fact that Q_A' is bornologic) may also be obtained by the

same methods as those used in the proof of Theorem 14 below; this method also provides us with another proof that $\mathcal{Q}_A'$ is bornologic.

Conversely, let M be a neighborhood of zero in $\mathcal{Q}_A'$ and denote by M the set of inverse Fourier transforms of the functions in M. By the definition of the topology of $\mathcal{Q}_A'$, we may assume that there is a bounded set K in $\mathcal{Q}_A$ such that M consists of all $W \in \mathcal{Q}_A'$ with $|W \cdot f| \leqq 1$ for all $f \in K$. Since K is bounded in $\mathcal{Q}_A$, for each n-tuple of integers $(l_1, l_2, \cdots, l_n)$ we can find a positive number $c_{l_1 l_2 \cdots l_n}$ such that every $f \in K$ satisfies

$$(63) \qquad |f(z)| \leqq c_{l_1 l_2 \cdots l_n} \exp(|z_1|^{A+1/l_1} + |z_2|^{A+1/l_2} + \cdots + |z_n|^{A+1/l_n}).$$

Using Cauchy's formula we can deduce easily that every $f \in K$ satisfies

$$(64) \qquad |f_{k_1 k_2 \cdots k_n}| (k_1/(A+1/l_1))^{(k_1/(A+1/l_1))} \cdots (k_n/(A+1/k_n))^{(k_n/(A+1/l_n))}$$
$$\leqq c_{l_1 l_2 \cdots l_n} \exp[(k_1/(A+1/l_1)) + (k_2/(A+1/l_2) + \cdots + (k_n/(A+1/l_n))]$$

for all integers $k_1, k_2, \cdots, k_n, l_1, l_2, \cdots, l_n$. (We may clearly assume that $c_{00\cdots 0} = 1$ and that $c_{l_1 l_2 \cdots l_n} \leqq c_{l_1' l_2' \cdots l_n'}$ if $l_1 \leqq l_1', l_2 \leqq l_2', \cdots, l_n \leqq l_n'$.)

Let $\{\gamma_j\}$ be a strictly increasing sequence of numbers with $\gamma_0 = 0$; let $\{d_j\}$ be a sequence of numbers with $d_0 = 0$, $d_j > 0$ for $j > 0$. We require that

(a) $$d_j + \gamma_j \leqq \gamma_{j+1} \text{ for any } j.$$

We define $\alpha_{j_1 j_2 \cdots j_n}$ as the region consisting of all $y \in C$ with

$$d_{j_k} + \gamma_{j_k} \leqq |y_k| \leqq \gamma_{j_k+1}$$

for all k. We require further that

(b) $$\exp(|y|^{A'-j_1'} + |y_2|^{A'-j_2'} + \cdots + |y_n|^{A'-j_n'})/c_{j_1+1, \cdots, j_n+1}$$
$$\geqq \exp(|y_1|^{A'-(j_1-1)'} + |y_2|^{A'-(j_2-1)'} + \cdots + |y_n|^{A'-(j_n-1)'})$$

whenever $|y_k| \geqq d_{j_k} + \gamma_{j_k}$ for all k and whenever $j_1 \geqq 1, \cdots, j_n \geqq 1$, where for any integer l, $A' - l'$ is defined by

$$(65) \qquad 1/(A+1/l)) + 1/(A'-l') = 1 - l^{d-1/2}$$

where $d < 0$ will be chosen later.

(c) There is no point of the form $[l/(A'-m')]^{-(A'-m')}$

in the interval $\gamma_j \leqq x \leqq \gamma_j + d_j$ *for* any j, where m, l are integers and $m \leqq 2(j+1)$.

The existences of sequences $\{\gamma_j\}$, $\{d_j\}$ satisfying (a), (b), and (c) is

not difficult to see. Moreover, it is clear that the regions $\alpha_{j_1 j_2 \cdots j_n}$ do not overlap.

We set

$$(66)\quad H(z) = \exp(|z_1|^{A'-j_1'} + |z_2|^{A'-j_2'} + \cdots + |z_n|^{A'-j_n'}/c_{j_1+1, j_2+1, \cdots, j_n+1}$$

for z in $\alpha_{j_1 j_2 \cdots j_n}$. The definition of H is completed by requiring that H be continuous and monotonic, i.e. $H(z) \geqq H(z')$ if $|z_1| \geqq |z_1'|, \cdots |z_n| \geqq |z_n'|$. It is obvious from (66) and (b) above that H satisfies (62).

Call N the set of all $G \in \boldsymbol{Q}_A'$ for which $|G(z)| \leqq H(z)$ for all $z \in C$; we claim $bN \subset M$ for some $b > 0$. Let $G \in N$ and let $(k_1, k_2, \cdots, k_n)$ be an n-tuple of non-negative integers. It is not difficult to see that we can find an n-tuple of integers $(l_1, l_2, \cdots, l_n)$ and an l-tuple of positive numbers $(a_1, a_2, \cdots, a_n)$ such that, for $h = 1, \cdots, n$,

$$(67)\qquad [k_h/(A' - l_h')]^{-(A'-l_h')} \geqq a_h \geqq [k_h/(A' - (l_h+1)')]^{-(A'-l_h')}$$

and such that the n-chain Ξ on C defined by

$$|z_1| = a_1, |z_2| = a_2, \cdots, |z_n| = a_n \text{ lies in } \alpha_{l_1 l_2 \cdots l_n}.$$

By Cauchy's formula we have [5]

$$(\boldsymbol{F}^{-1}G)_{k_1 k_2 \cdots k_n} = -ik_1!k_2! \cdots k_n! \int G(z)/z_1^{k_1+1} z_2^{k_2+1} \cdots z_n^{k_n+1}\, dz.$$

Now, on Ξ we have

$$|G(z)| \leqq H(z) = \exp(|z_1|^{A'-l_1'} + |z_2|^{A'-l_2'} + \cdots + |z_n|^{A'-l_n'})/c_{l_1+1, l_2+1, \cdots, l_n+1}$$

because of (66). Using (67) we find

$$|(\boldsymbol{F}^{-1}G)_{k_1 k_2 \cdots k_n}|\, c_{l_1+1, l_2+1, \cdots, l_n+1} [k_1/(A' - (l_1+1)')]^{k_1(A'-l_1')} \times \cdots [k_n/(A' - (l_n+1)')]^{k_n/(A'-l_n')}$$

$$\leqq k_1!k_2! \cdots k_n! \exp(k_1/(A' - l_1') + \cdots k_n/(A' - l_n')).$$

By Stirling's formula and (65) we can find a constant $\theta > 0$ such that

$$(68)\quad (\boldsymbol{F}^{-1}G)_{k_1 k_2 \cdots k_n}\, c_{l_1+1, l_2+1, \cdots, l_n+1}$$

$$\leqq \theta \exp(-k_1/(A + 1/l_1) - k_2/(A + 1/l_2) - \cdots - k_n/(A + 1/l_n))$$

$$\times k_1^{k_1/(A+1/l_1)}\, k_2^{k_2/(A+1/l_2)} \cdots k_n^{k_n/(A+1/l_n)}$$

$$\times \exp(-k_1 - k_2 - \cdots - k_n) A'^{k_1/(A'-1) + k_2/(A'-1) + \cdots + k_n/(A'-1)}.$$

Now, use (49), (64), (65) and (68); it is clear that we can find $d < 0$ so small that, for some θ', for all $G \in N$, we have

$$|F^{-1}G\cdot f|\leqq\theta' \text{ for all } f\in K. \tag{69}$$

This proves that $(1/\theta')N\subset M$ which is the desired result.

We can now use the method of proof of Theorem 8 (except that we use Theorem 2 of [4], p. 323 in place of Theorem 5, p. 317) to deduce

THEOREM 12. *Let $J\in\boldsymbol{Q}_A'$, $J\neq 0$; then $JG\to G$ is a continuous linear map of $J\boldsymbol{Q}_A'\to\boldsymbol{Q}_A'$.*

From this we deduce easily

COROLLARY. *For any $W\in\mathfrak{Q}_A'$, $W\neq 0$, $W*U\to U$ is a continuous linear map of $W*\mathfrak{Q}_A'$ into $\mathfrak{Q}_A'$.*

THEOREM 13. *For any $W\in\mathfrak{Q}_A'$, $W*\mathfrak{Q}_A=\mathfrak{Q}_A$.*

Remark. Theorem 13 can also be derived from Theorem 4 of [4], p. 323 by use of certain *onto* theorems for Frechet spaces (see [1]). However, this method uses strongly the fact that $\mathfrak{Q}_A$ is metrizable and so does not apply to more general situations.

6. The ring of formal power series. Let $\mathcal{O}$ denote the ring of formal power series in n variables; that is, each $f\in\mathcal{O}$ may be written in the form $\sum f_{j_1j_2\cdots j_n}Z_1^{j_1}Z_2^{j_2}\cdots Z_n^{j_n}$. The topology of $\mathcal{O}$ is that of convergence of each coefficient. It is readily verified that $\mathcal{O}$ is a complete, Montel, bornologic, reflexive, topological vector space which is metrizable, but $\mathcal{O}$ is not a Schwartz space. The dual $\mathcal{O}'$ of $\mathcal{O}$ (with the topology of uniform convergence on the bounded sets of $\mathcal{O}$) may be identified with the space of partial differential operators; for each $U\in\mathcal{O}'$ we write

$$U=\sum(j_1!\cdots j_n!)^{-1}U_{j_1\cdots j_n}\partial Z_1^{j_1+\cdots+j_n}/\partial Z_1^{j_1}\cdots\partial Z_n^{j_n}$$

(finite sum). Then we have

$$U\cdot f=\sum U_{j_1j_2\cdots j_n}f_{j_1j_2\cdots j_n}. \tag{70}$$

The convolutions $U*f$, $U*W$ for U, $W\in\mathcal{O}'$, $f\in\mathcal{O}$ are defined by requiring that the analog of Proposition 11 should hold for $\mathcal{O}$; we have continuity properties similar to those in Propositions 9, 10 above.

The Fourier transform on $\mathcal{O}'$ is defined as for the spaces $\mathcal{S}'$, $\mathfrak{Q}_A'$. $\boldsymbol{O}'=\boldsymbol{F}\mathcal{O}'$ may be identified with the space of polynomials in n variables. For any $U\in\mathcal{O}'$, we have

$$\boldsymbol{F}(U)=\sum U_{j_1j_2\cdots j_n}Z_1^{j_1}Z_2^{j_2}\cdots Z_n^{j_n}. \tag{71}$$

The topology of $\boldsymbol{O}'$ is of interest because it is a "natural" topology for the space of polynomials.

THEOREM 14. *Let H be any continuous function on C such that, for any j, p, we have*

$$|z_j|^p = O(H(z)). \tag{72}$$

Call N the set of polynomials P such that $|P(z)| \leqq H(z)$ for all $z \in C$; the sets N form a fundamental system of neighborhoods of zero in $\boldsymbol{O}'$.

Proof. Let H and N satisfy the hypotheses of Theorem 14; we want to show that N is a neighborhood of zero in $\boldsymbol{O}'$. For each n-tuple of non-negative integers $(k_1, k_2, \cdots, k_n)$, let $a_{k_1 k_2 \cdots k_n}$ be any positive number such that

$$a_{k_1 k_2 \cdots k_n} |z_1|^{k_1} |z_2|^{k_2} \cdots |z_n|^{k_n} \leqq 2^{-k_1-k_2-\cdots-k_n} H(z) 2^{-n} \tag{73}$$

for all $z \in C$. Let B be the set of all $f \in \mathfrak{O}$ for which

$$|f_{k_1 k_2 \cdots k_n}| \leqq a_{k_1 k_2 \cdots k_n} \tag{74}$$

for all $k_1, k_2, \cdots, k_n$. Call M the set of $U \in \mathfrak{O}'$ for which $|U \cdot f| \leqq 1$ for all $f \in B$; we claim that $U \in M$ implies $\boldsymbol{F}(U) \in N$.

For, given any $k_1, k_2, \cdots, k_n$, we have

$$Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n} \in (a_{k_1 k_2 \cdots k_n})^{-1} B; \text{ so that } |U_{k_1 k_2 \cdots k_n}| \leqq a_{k_1 k_2 \cdots k_n}$$

or, for any $z \in C$,

$$\begin{aligned} |U(z)| &\leqq |U_{k_1 k_2 \cdots k_n}| \, |z_1|^{k_1} |z_2|^{k_2} \cdots |z_n|^{k_n} \\ &\leqq \sum a_{k_1 k_2 \cdots k_n} |z_1|^{k_1} |z_2|^{k_2} \cdots |z_n|^{k_n} \leqq \sum 2^{-k_1-k_2-\cdots-k_n} H(z) 2^{-n} \leqq H(z). \end{aligned}$$

Thus, $\boldsymbol{F}(U) \in N$ which proves that N is a neighborhood of zero in $\boldsymbol{O}'$.

Call λ the topology induced by the sets N satisfying the hypotheses of Theorem 14 on the set of functions of $\boldsymbol{O}'$. Then it is clear that λ is a locally convex topology; by the above, λ is weaker than the topology of $\boldsymbol{O}'$. Now, it is readily verified that the bounded sets of λ and $\boldsymbol{O}'$ are the same, namely, the sets B of polynomials for which we can find a constant $A > 0$ such that every $P \in B$ satisfies $|P(z)| \leqq A(1 + |z|^A)$ for all $z \in C$. Thus, in order to show that the topology λ is the same as that of $\boldsymbol{O}'$, it is sufficient to show that λ defines a bornologic space. (Because it is clear that, among all locally convex topologies that can be assigned to a given vector space such that a given family of sets shall be the bounded sets, the bornologic topology is the strongest.)

Let M be any convex set in $\boldsymbol{O}'$ which swallows every bounded set in

$\boldsymbol{O}'$; we have to show that M is a neighborhood of zero for λ. For each $k_1, k_2, \cdots, k_n$, the set

$$S_{k_1 k_2 \cdots k_n} = \{\epsilon Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}\}_{|\epsilon| \leq 1} \tag{75}$$

is bounded in $\boldsymbol{O}'$; thus, we can find a $b_{k_1 k_2 \cdots k_n} > 0$ with

$$b_{k_1 k_2 \cdots k_n} S_{k_1 k_2 \cdots k_n} \subset M. \tag{76}$$

Let us define the function H on C as follows: We choose a strictly increasing sequence β_k of non-negative numbers, with $\beta_0 = 0$, and a sequence of non-negative numbers d_k such that $d_0 = 0$, $d_k > 0$ for $k > 0$, $d_k + \beta_k < \beta_{k+1}$ for all k, and

$$|y_1^{k_1} y_2^{k_2} \cdots y_n^{k_n} b_{k_1 k_2 \cdots k_n}| \leqq |y_1^{k_1+1} y_2^{k_2+1} \cdots y_n^{k_n+1} 2^{-n-k_1-k_2-\cdots-k_n}| \tag{77}$$

whenever $|y_j| \geqq d_j + \beta_j$ for all j. We call $\alpha_{k_1 k_2 \cdots k_n}$ the region defined by the relations

$$\alpha_{k_1 k_2 \cdots k_n}\colon\ \beta_{k_1} + d_{k_1} \leqq |z_1| \leqq \beta_{k_1+1}, \cdots, \beta_{k_n} + d_{k_n} \leqq |z_n| \leqq \beta_{k_n+1}. \tag{78}$$

Next, set

$$H(z) = 2^{-n-k_1-k_2-\cdots-k_n} |z_1|^{k_1} |z_2|^{k_2} \cdots |z_n|^{k_n} b_{k_1 k_2 \cdots k_n} \text{ in } \alpha_{k_1 k_2 \cdots k_n}. \tag{79}$$

The definition of H is completed by requiring that H be continuous and monotonic.

It is clear from (77) and (78) that H satisfies (72). Thus, the set N of $P \in \boldsymbol{O}'$ for which $|P(z)| \leqq H(z)$ is a neighborhood of zero for λ; we claim $N \subset M$. For, write $P \in N$ in the form

$$P = \sum P_{k_1 k_2 \cdots k_n} Z_1^{k_1} Z_2^{k_2} \cdots Z_n^{k_n}. \tag{80}$$

Given any n-tuple of non-negative integers $(l_1, l_2, \cdots, l_n)$, we can find positive numbers $a_{l_1}, a_{l_2}, \cdots, a_{l_n}$ such that the n-chain Ξ defined by: $|z_1| = a_{l_1}, |z_2| = a_{l_2}, \cdots, |z_n| = a_{l_n}$ lies in $\alpha_{l_1 l_2 \cdots l_n}$. By Cauchy's formula we may write [5]

$$P_{l_1 l_2 \cdots l_n} = i \int_{\Xi} P(z)/z_1^{l_1+1} z_2^{l_2+1} \cdots z_n^{l_n+1}\, dz.$$

Thus, since $|P(z)| \leqq H(z)$ for all z,

$$|P_{l_1 l_2 \cdots l_n}| \leqq H(a_{l_1}, a_{l_2}, \cdots a_{l_n})/a_{l_1}^{l_1} a_{l_2}^{l_2} \cdots a_{l_n}^{l_n} = b_{l_1 l_2 \cdots l_n} 2^{-n-l_1-l_2-\cdots-l_n}$$

by (79). Thus, by (75) and (76), we have

$$P_{l_1 l_2 \cdots l_n} Z_1^{l_1} Z_2^{l_2} \cdots Z_n^{l_n} \in 2^{-n-l_1-l_2-\cdots-l_n} M.$$

Since M is convex, we have $P \in M$; thus, $N \subset M$ which shows that λ is bornologic.

In the course of the proof of Theorem 14 we have shown

PROPOSITION 13. *$\mathcal{O}'$ is bornologic.*

By the method of proof of Theorem 8 (except that we use Lemma 3 of [2], p. 890 instead of Theorem 5 of [4], p. 317) we deduce

THEOREM 15. *For any polynomial P, $PG \to G$ is a continuous linear map of $P\mathcal{O}'$ onto $\mathcal{O}'$.*

We obtain easily

COROLLARY. *For each $W \in \mathcal{O}'$, $W * U \to U$ is a continuous linear map of $W * \mathcal{O}'$ onto $\mathcal{O}'$.*

THEOREM 16. *For any $W \in \mathcal{O}'$, $W * \mathcal{O}' = \mathcal{O}'$.*

Remark. Theorem 16 may also be derived easily from Theorem 1 of [4], p. 322 by use of certain onto-theorems for Frechet spaces (see [1]).

7. General remarks.

1. The methods used in Section 5 may be extended to other spaces for which the Fourier transform gives rise to entire functions. Suppose that Λ is a set and, for each $\lambda \in \Lambda$ we are given a pair of functions ϕ_λ, ψ_λ which are conjugate in the sense of Young (see [15]). Then, under suitable conditions, if we define the space $\mathcal{Q}_\Phi$ as consisting of all entire functions f on C such that

$$f(z) = O[\exp(\phi_\lambda(z))] \text{ for all } \lambda \in \Lambda \tag{81}$$

with the topology defined by means of the semi-norms

$$v_\lambda(f) = \sup_{z \in C} |\exp(-\phi_\lambda(z)) f(z)| \tag{82}$$

then we find that the space of all entire functions G on C such that

$$G(z) = O[\exp(\psi_\lambda(z))] \text{ for some } \lambda \in \Lambda \tag{83}$$

turns out to be the space $\mathcal{Q}_\Phi'$ which is the Fourier transform of the dual of $\mathcal{Q}_\Phi'$. The topology of $\mathcal{Q}_\Phi'$ may be studied by means of the methods of Sections 4, 5, and 6, and we may also study the division problem in $\mathcal{Q}_\Phi$.

The methods of Sections 4, 5, 6 may also be extended to other kinds of spaces, e. g. the space $\mathcal{B}$ of functions analytic at the origin. (The topology

of $\mathcal{B}$ is defined in a natural manner in [7].) In this case, the Fourier transform $\boldsymbol{B}'$ of the dual space is the space of all entire functions G such that

$$G(z) = O[\exp(\epsilon |z|)] \text{ for every } \epsilon > 0. \tag{84}$$

We can establish the analog of the theorems of Sections 4, 5, 6 for $\mathcal{B}$.

By taking other subspaces of $\mathcal{O}$ we obtain interesting classes of spaces of entire functions of order $\leqq 1$. These spaces are, in general, not Frechet spaces, so we cannot use the onto-theorems of [1] to prove the analog of Theorems 9, 13, and 16, but we must use the methods of proof of these theorems as given in this paper.

2. The results of Sections 2 and 3 can also be extended to other spaces. For example, the space $\mathcal{D}_\times$ which is the space of continuous linear maps of $\mathcal{D}'$ into $\mathcal{D}$ with the compact-open topology (see my paper, "On the theory of kernels of Schwartz," to appear in *Proceedings of the American Mathematical Society*). The elements of $\mathcal{D}_\times$ are the indefinitely differentiable functions of compact carrier on $R \times R$; the topology of $\mathcal{D}_\times$ is described as follows: For each integer $l \geqq 0$ we choose a finite sequence of differential operators $Q_1^l, Q_2^l, \cdots, Q_{l_l}^l$ on R and a number $b_l > 0$. Then we consider the set N of all $f \in \mathcal{D}_\times$ such that, if $(x, y) \in R \times R$, $|x| \geqq l_1$, $|y| \geqq l_2$,

$$|(Q_{k,2}{}^{l_1} Q_{k',2}{}^{l_2} f)(x, y)| \leqq b_{l_1} b_{l_2} \tag{85}$$

for $k = 1, 2, \cdots, l_{1j}$, $k' = 1, 2, \cdots, l_{2j}$, where $Q_{k,1}{}^{l_1}$ means that the operator $Q_k{}^{l_1}$ operates on x, and $Q_{k',2}{}^{l_2}$ is defined similarly for y.

The topology of the Fourier transform $\boldsymbol{D}_\times$ of $\mathcal{D}_\times$ may be described in a manner similar to that in Theorem 1, except that the regions $\Gamma_l{}^k$ which appeared there have to be replaced by regions of the form $\Gamma_l{}^k \times \Gamma_l{}^{k'}$.

3. The question remains as to what is the most general distribution of compact carrier W for which $W * \mathcal{D}' = \mathcal{D}'$, or $W * \mathcal{E} = \mathcal{E}$. We are led to a class of distributions whose Fourier transform is "slowly decreasing." This question will be dealt with in a future paper. (See "Completely inversible operators," *Proceedings of the National Academy of Science*, vol. 41 (1955), pp. 945-946.)

4. We have, in this paper, been concerned solely with complex-valued functions. It is also possible to define analogues of the spaces considered in this paper for vector-valued functions. The Fourier transform is defined as in [5]. There is no essential difficulty in describing the Fourier transform of the dual spaces in any case except for the analog of the space $\mathcal{D}$. A

description of the Fourier transform of this space in general, and even in certain particular cases seems to be extremely difficult.

5. The question naturally arises as to what extent the results of this paper can be extended to an arbitrary Lie group G in place of R or C. If G is compact, the results are very simple to obtain. The next "attackable" case is for G semi-simple. The methods of the author and F. I. Mautner (see "Some properties of the Fourier transform on semi-simple Lie groups I," *Annals of Mathematics*, vol. 61 (1955), pp. 406-439; see also the sequel to that paper which is forthcoming) together with the methods of this paper apply to certain semi-simple Lie groups. The case of the general, or solvable, Lie group seems to be very difficult.

THE INSTITUTE FOR ADVANCED STUDY.

REFERENCES.

[1] J. Dieudonné and L. Schwartz, "La dualité dans les espaces $(\mathcal{F})$ et $(\mathcal{LF})$," *Annales de l'Institut Fourier* (Grenoble), vol. I (1950), pp. 61-101.

[2] L. Ehrenpreis, "Solutions of some problems of division I," *American Journal of Mathematics*, vol. 76 (1954), pp. 883-903.

[3] ———, "Solutions of some problems of division II," *American Journal of Mathematics*, vol. 77 (1955), pp. 286-292.

[4] ———, "Mean periodic functions I," *American Journal of Mathematics*, vol. 77 (1955), pp. 293-328.

[5] ———, "Analytic functions and the Fourier transform of distributions I," *Annals of Mathematics*, vol. 63 (1956), pp. 129-159.

[6] ———, "Analytic functions and the Fourier transform of distributions II" (to appear).

[7] ———, *Theory of Infinite Derivatives* (to appear).

[8] ———, "The division problem for distributions," *Proceedings of the National Academy of Sciences*, vol. 41 (1955), pp. 756-758.

[9] A. Grothendieck, "Sur les espaces $(\mathcal{F})$ et $(\mathcal{DF})$," to appear in *Summa Brasiliensis Mathematicae*.

[10] ———, *Products Tensoriels Topologiques et Espaces Nucléaires*, Memoirs of the American Mathematical Society, No. 16, 1955.

[11] B. Malgrange, "Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution," thesis, Paris, 1955, to appear in *Annales de l'Institut Fourier* (Grenoble).

[12] L. Schwartz, *Théorie des Distributions*, vol. I-II, Paris, 1950-51.

[13] E. C. Titchmarsh, *The Theory of Functions*, Oxford, 1932.

[14] ———, *Theory of Fourier Integrals*, Oxford, 1937.

[15] A. Zygmund, *Trigonometric Series*, Warsaw, 1935.

CHARACTERISTIC LINEAR SYSTEMS OF COMPLETE CONTINUOUS SYSTEMS.*

By K. KODAIRA.†

Let V be a non-singular projective variety and let $\mathfrak{M}$ be a complete continuous system i. e., a maximal algebraic system of effective divisors D on V. Then, letting M be the parameter variety of $\mathfrak{M}$, we have a one-to-one algebraic correspondence $\lambda \to D = D_\lambda$ between M and $\mathfrak{M}$. Suppose that a member $C = D_o$ of $\mathfrak{M}$ is a non-singular prime divisor and that the corresponding point o is a simple point of M. Then the characteristic linear system of $\mathfrak{M}$ on C can be defined in a well known manner.[1] Roughly speaking, the characteristic linear system of $\mathfrak{M}$ on C is the set of all divisors $D_\lambda \cdot C$ on C cut out by divisors D_λ belonging to $\mathfrak{M}$ which are infinitely near to $C = D_o$. The main purpose of the present paper is to establish *the completeness of the characteristic linear system*[2] *of* $\mathfrak{M}$ *on* C under an additional restriction[3] on C. Let $\Omega(C)$ be the sheaf over V of germs of meromorphic functions which are multiples of $-C$ and denote by $H^1(V, \Omega(C))$ the first cohomology group of V with coefficients in $\Omega(C)$. Then our main results can be stated as follows:

THEOREM.[4] *Let C be a non-singular prime divisor belonging to a complete continuous system $\mathfrak{M}$ of effective divisors on a non-singular projective variety V. If C satisfies the condition*

$$\text{(a)} \qquad H^1(V, \Omega(C)) = 0,$$

---

* Received February 14, 1956.

† This work was supported by a research project at Princeton University sponsored by the Office of Ordnance Research, U. S. Army.

[1] See Zariski [19], p. 78.

[2] For classical results concerning complete continuous systems of curves on algebraic surfaces, see Zariski [19], Chap. V.

[3] Zappa has exhibited an example of a complete continuous system of curves on an algebraic surface whose characteristic linear system on its general member is incomplete. It is therefore necessary to impose some additional restrictions on C in order to exclude such exceptional cases. See Zappa [20].

[4] In case V is an algebraic surface, this theorem reduces to a special case of a theorem of Severi to the effect that the characteristic linear system of $\mathfrak{M}$ on C is complete if $|C|$ is "semi-regular." See Severi [17].

then C corresponds to a simple point of the canonical parameter variety M of $\mathfrak{M}$ and the characteristic linear system of $\mathfrak{M}$ on C is complete. Moreover, the complete continuous system $\mathfrak{M}$ containing C is uniquely determined by C.

Let K be the canonical divisor on V. Then it can be shown that C *satisfies the condition* (a) *if*

(c') $|m(C-K)|$ *is ample*[5] *for sufficiently large integer* m.

Hence it is a corollary of the above theorem that the characteristic linear system of the complete continuous system $\mathfrak{M}$ on its member C is complete if C belongs to the adjoint system $|K+E|$ of a system $|E|$ whose multiple $|mE|$ is ample for a sufficiently large integer[6] m.

We note that we are concerned with the *classical case* i.e. the case in which the varieties are defined over the field of complex numbers. In case the ground field is of characteristic $p \neq 0$, the theorem of the completeness of the characteristic linear systems of complete continuous systems is false, as was shown recently by J. Igusa.[7]

1. Preliminaries.[8] Let V be a non-singular algebraic variety of complex dimension n imbedded in a projective space. By a *complex line bundle* F over V we shall mean an analytic fibre bundle over V whose fibre is a complex line $\boldsymbol{C}$ and whose structure group is the multiplicative group $\boldsymbol{C}^*$ of complex numbers acting on $\boldsymbol{C}$. The bundle F may be described as follows: Let $\{U_j\}$ be a sufficiently fine finite covering of V and let π be the canonical projection of F onto V. Then the inverse image $\pi^{-1}(U_j)$ has a product structure: $\pi^{-1}(U_j) = U_j \times \boldsymbol{C}$, and $(z, \zeta_j) \in U_j \times \boldsymbol{C}$ is identical with $(z, \zeta_k) \in U_k \times \boldsymbol{C}$ if and only if $\zeta_j = f_{jk}(z) \cdot \zeta_k$, where $f_{jk}(z)$ is a non-vanishing holomorphic function defined in $U_j \cap U_k$. Under these circumstances, we say that the bundle F is defined by the system $\{f_{jk}\}$ of the *transition functions* $f_{jk} = f_{jk}(z)$, and we call ζ_j the *fibre coordinate* of the point (z, ζ_j) on F over the neighborhood U_j. We identify two complex line bundles which are

[5] We say that a complete linear system $|E|$ on V is ample if $|E|$ coincides with the system of section of V cut out by hyperplanes in one of its ambient projective spaces.

[6] This special case of our main theorem has been announced by the author at the International Congress of Mathematicians 1954. Under the stronger assumption that $|C|$ is sufficiently ample, the author has proved the completeness of $\mathfrak{M}$ on C by means of the theory of harmonic integrals. See Kodaira [8], § 11.

[7] Igusa [7a].

[8] In this Section we give a brief summary of some known results concerning complex line bundles and analytic sheaves. Cf. Kodaira [9], [10], Kodaira and Spencer [12], [13].

analytically equivalent. For any pair of bundles E, F determined respectively by $\{e_{jk}\}$, $\{f_{jk}\}$, we define the sum $E+F$ to be the bundle determined by $\{e_{jk}\cdot f_{jk}\}$. Then *the set* $\mathfrak{F}=\{F\}$ *of all complex line bundles over* V *forms an additive group.*

For any divisor D on V defined in each U_j by a local meromorphic function $R_j(D)=R_j(z;D)$, we denote by $[D]$ the line bundle over V defined by the system $\{f_{jk}(D)\}$ of the functions $f_{jk}(D)=R_j(D)/R_k(D)$. It is obvious that $[D]$ coincides with $[D']$ if and only if D is linearly equivalent to D'. Given a bundle $F\in\mathfrak{F}$, *the set of all effective divisors* D *satisfying* $[D]=F$ *forms therefore a complete linear system on* V. We denote this complete linear system by the symbol $|F|$. Obviously $|[D]|$ *coincides with* $|D|$. In view of this, we write moreover $|D+F|$ for $|[D]+F|$.

We denote by $\Gamma(F)$ the linear space consisting of all holomorphic sections of F over V. A holomorphic section $\phi\in\Gamma(F)$ is, by definition, a holomorphic mapping $z\to\phi(z)$ of V into F satisfying $\pi\phi(z)=z$. Letting $\phi_j(z)$ be the fibre coordinate of $\phi(z)$ over U_j, we infer from $\phi_j(z)=f_{jk}(z)\phi_k(z)$ that the divisor (ϕ_j) of the holomorphic function $\phi_j(z)$ coincides with the divisor (ϕ_k) of $\phi_k(z)$ in $U_j\cap U_k$. Hence we may define *the divisor* (ϕ) *of the section* ϕ by setting $(\phi)=(\phi_j)$ on each U_j, provided that $\phi\not\equiv 0$. As one readily infers, the complete linear system $|F|$ consists of all divisors $D=(\phi)$, $\phi\in\Gamma(F)$:

$$|F|=\{(\phi)\,|\,\phi\in\Gamma(F),\not\equiv 0\}. \tag{1.1}$$

Obviously (ψ) coincides with (ϕ) if and only if $\psi=c\cdot\phi$, $c\in\boldsymbol{C}^*$. Consequently we obtain

$$\dim|F|=\dim\Gamma(F)-1. \tag{1.2}$$

By the *characteristic class* $c(F)$ of a complex line bundle F over V we shall mean the characteristic class of the principal bundle associated with F. The characteristic class $c(F)$ is an element of the second cohomology group $H^2(V,\boldsymbol{Z})$ of V with coefficients in the integers $\boldsymbol{Z}$. Let $\boldsymbol{R}$ be the reals. For any element $c\in H^2(V,\boldsymbol{Z})$ we denote by $c_{\boldsymbol{R}}$ the element of $H^2(V,\boldsymbol{R})$ corresponding to c under the homomorphism $H^2(V,\boldsymbol{Z})\to H^2(V,\boldsymbol{R})$ induced by the inclusion map $\boldsymbol{Z}\to\boldsymbol{R}$. In view of de Rham's theorem, $c_{\boldsymbol{R}}$ can be regarded as a class of closed real 2-forms on V.

THEOREM 1.1.[9] *Let* F *be the complex line bundle over* V *defined by a system* $\{f_{jk}\}$ *of transition functions* f_{jk} *with respect to a covering* $\{U_j\}$,

[9] Kodaira [10], p. 1271, Lemma.

and let $\{a_j\}$ *be a system of positive functions* a_j *defined respectively on* U_j *satisfying* $a_j = |f_{jk}|^2 a_k$ *in* $U_j \cap U_k$. *Then* $\gamma = (\frac{i}{2\pi})\partial\bar{\partial}\log a_j$ *is a closed real* $(1,1)$*-form on* V *belonging to the characteristic class* $c_R(F)$ *of the bundle* F.

We say that a real $(1,1)$-form $\gamma = i\sum \gamma_{\alpha\bar{\beta}}(z,\bar{z})dz_\alpha d\bar{z}_\beta$ on V is *positive* and write $\gamma > 0$ if the Hermitian form $\sum \gamma_{\alpha\bar{\beta}}(z,\bar{z})u_\alpha\bar{u}_\beta$ in n variables $u_1, \cdots, u_n$ is positive definite at each point z on V. Moreover, we say that a cohomology class $c \in H^2(V, \boldsymbol{Z})$ is positive and write $c > 0$ if c_R contains a closed real $(1,1)$-form $\gamma > 0$. By the canonical bundle K over V we shall mean the complex line bundle of $(n,0)$-forms over V, where n is the dimension of V. Now we denote by $\Omega(F)$ the sheaf of germs of holomorphic sections of the bundle F. Then we have

THEOREM 1.2.[10] *The cohomology group* $H^q(V, \Omega(F))$ *vanishes for* $q \geqq 1$ *if the characteristic class* $c(F-K)$ *of* $F-K$ *is positive.*

We say that a complete linear system $|D|$ on V is *ample* if there exists an ambient projective space $\mathfrak{S}$ of V such that $|D|$ coincides with the system of all sections of V cut out by the hyperplanes in $\mathfrak{S}$.

THEOREM 1.3.[11] *There exists on* V *a real* $(1,1)$*-form* γ_0 *such that the complete linear system* $|F|$ *is ample for any complex line bundle* F *whose characteristic class* $c_R(F)$ *contains a closed real* $(1,1)$*-form* $\gamma > \gamma_0$.

Combining this with the obvious fact that $c(F)$ is positive if $|F|$ is ample, we infer that the *characteristic class* $c(F)$ *of* F *is positive if and only if the complete linear system* $|mF|$ *is ample for sufficiently large positive integer* m.

Let $\mathfrak{A}$ be the linear space over $\boldsymbol{C}$ consisting of all simple differentials of the first kind on V. The first cohomology group $H^1(V, \boldsymbol{Z})$ is free abelian. Let $\{b_1, \cdots, b_r, \cdots, b_{2q}\}$ be a base of $H^1(V, \boldsymbol{Z})$. Then, for each b_r, there exists one and only one element β_r of $\mathfrak{A}$ such that

$$2\pi i b_r[Z] = \int_Z (\bar{\beta}_r - \beta_r) \tag{1.3}$$

for any integral 1-cycle Z on V. Clearly $\{\beta_1, \cdots, \beta_r, \cdots, \beta_{2q}\}$ forms a base of $\mathfrak{A}$ with respect to the reals $\mathbf{R}$. We denote by $\mathfrak{d}$ the discrete subgroup of $\mathfrak{A}$ generated by $\beta_1, \cdots, \beta_r, \cdots, \beta_{2q}$. Now let $\mathfrak{P}$ be the subgroup of $\mathfrak{F}$

[10] Kodaira [11], Theorem 3.
[11] Kodaira [11], Theorem 3.

consisting of all complex lines bundles P with $c(P)=0$. Then we have the exact sequence [12]

$$0 \to \bar{\mathfrak{d}} \xrightarrow{i} \bar{\mathfrak{A}} \xrightarrow{\rho} \mathfrak{P} \to 0, \tag{1.4}$$

where i denotes the inclusion map. The homomorphism ρ is described explicitly as follows: For $\bar{\alpha} \in \bar{\mathfrak{A}}$, the image $P=\rho(\bar{\alpha})$ is the complex line bundle over V defined with respect to the covering $\{U_j\}$ by the system $\{\kappa_{jk}\}$ of the constants

$$\kappa_{jk} = \exp \int_{z(j)}^{z(k)} \bar{\alpha}, \tag{1.5}$$

where $z(j)$ is a point in U_j and $\int_{z(j)}^{z(k)}$ denotes the integral along a smooth curve *in* $U_j \cup U_k$ combining $z(j)$ with $z(k)$. The canonical complex structure on $\mathfrak{P}$ is obtained by identifying $\mathfrak{P}$ with the complex torus $\bar{\mathfrak{A}}/\bar{\mathfrak{d}}$ by means of the isomorphism $\bar{\mathfrak{A}}/\bar{\mathfrak{d}} \cong \mathfrak{P}$ induced by ρ. The complex manifold $\mathfrak{P}=\bar{\mathfrak{A}}/\bar{\mathfrak{d}}$ is called the *Picard variety* attached to V. It is well known that the *Picard variety* $\mathfrak{P}$ *is an algebraic variety imbedded in a projective space.*[13]

2. Construction of a complex line bundle. We form the product variety $V \times \mathfrak{P}$ and denote for any point $P \in \mathfrak{P}$ the mapping $z \to (z, P)$ of V into $V \times \mathfrak{P}$ by r_P. In this Section we construct a complex line bundle Ξ over $V \times \mathfrak{P}$ such that, for each complex line bundle $P \in \mathfrak{P}$, the bundle $r_P^*(\Xi)$ over V induced by the mapping r_P coincides with P: $r_P^*(\Xi)=P$.

Let M be a compact complex manifold, $\hat{M}$ the universal covering of M, and let Δ be the covering transformation group of $\hat{M}$ with respect to M. Moreover, let $f(\hat{z}, \tau)$, $\hat{z} \in \hat{M}$, $\tau \in \Delta$, be a non-vanishing complex-valued function which is holomorphic in $\hat{z}$ and satisfies

$$f(\hat{z}, \sigma\tau) = f(\tau\hat{z}.\sigma) \cdot f(\hat{z}, \tau), \qquad \sigma, \tau \in \Delta. \tag{2.1}$$

Then, defining for each $\tau \in \Delta$ an analytic automorphism $T(\tau)$ of the product variety $\hat{M} \times \boldsymbol{C}$ by

$$T(\tau): (\hat{z}, \zeta) \to (\tau\hat{z}, f(\hat{z}, \tau)\zeta), \tag{2.2}$$

we infer readily that $T(\sigma\tau^{-1}) = T(\sigma)T(\tau)^{-1}$ for $\sigma, \tau \in \Delta$. Thus

$$T(\Delta) = \{T(\tau) \mid \tau \in \Delta\}$$

---

[12] Kodaira and Spencer [12], p. 871.

[13] Lefschetz [15], pp. 364-370; see also Kodaira [11], p. 40.

forms a discontinuous group of analytic automorphisms of $M \times \boldsymbol{C}$ which is isomorphic to Δ, and the factor space $F = M \times \boldsymbol{C}/T(\Delta)$ is a complex manifold (without singularities). Obviously each automorphism $T(\tau)$ commutes with the canonical projection $\hat{\pi}: (\hat{z}, \zeta) \to \hat{z}$ of $\hat{M} \times \boldsymbol{C}$ onto $\hat{M}$. Hence $\hat{\pi}$ induces a holomorphic mapping π of F onto $M = \hat{M}/\Delta$, and moreover, for a sufficiently small open subset $U \subset M$, the inverse image $\pi^{-1}(U)$ has a canonical product structure $\pi^{-1}(U) = U \times \boldsymbol{C}$. Thus we infer that F is a complex line bundle[14] over M.

We remark that a system $\{f_{jk}\}$ of transition functions defining the bundle F can be obtained in the following manner: Denote by ϖ the canonical projection of $\hat{M}$ onto M. Letting $\{U_j\}$ be a finite covering of M by sufficiently small "spherical" neighborhoods U_j, we take, for each U_j, a spherical neighborhood $\hat{U}_j$ on $\hat{M}$ which is mapped by ϖ homeomorphically onto U_j, and denote by ϖ_j the restriction of ϖ to $\hat{U}_j$. Then, for each pair (j, k) such that $U_j \cap U_k$ is not empty, the mapping $\varpi_j^{-1}\varpi_k$ defined on $\hat{U}_k \cap \varpi^{-1}(U_j)$ coincides with an element τ_{jk} of Δ:

$$\varpi_j^{-1}\varpi_k = \tau_{jk}, \qquad \text{on } \hat{U}_k \cap \varpi^{-1}(U_j).$$

Now, setting

$$f_{jk}(z) = f(\varpi_k^{-1}(z), \tau_{jk}), \qquad \text{for } z \in U_j \cap U_k, \tag{2.3}$$

we infer readily that $\{f_{jk}(z)\}$ is a system of transition functions defining the complex line bundle $F = \hat{M} \times \boldsymbol{C}/T(\Delta)$. In fact, denote by $[\hat{z}, \zeta]$ the point on F corresponding to $(\hat{z}, \zeta) \in \hat{M} \times \boldsymbol{C}$. Then each point η on the fibre $\pi^{-1}(z)$, $z \in U_j$, can be written uniquely in the form $\eta = [\varpi_j^{-1}(z), \zeta_j]$. Thus we may use ζ_j as the fibre coordinate of η over U_j. Now, in view of (2.2), we have

$$[\varpi_k^{-1}(z), \zeta_k] = [\tau_{jk}\varpi_k^{-1}(z), f(\varpi_k^{-1}(z), \tau_{jk})\zeta_k]$$

and therefore, by (2.3),

$$[\varpi_k^{-1}(z), \zeta_k] = [\varpi_j^{-1}(z), f_{jk}(z)\zeta_k] \qquad \text{for } z \in U_j \cap U_k.$$

This proves that the law of transition of fibre coordinates is given by $\zeta_j = f_{jk}(z)\zeta_k$.

LEMMA 2.1. *Let $a(\hat{z})$ be a positive function of class C^∞ on M satisfying*

$$a(\tau\hat{z}) = |f(\hat{z}, \tau)|^2 \cdot a(\hat{z}), \quad \tau \in \Delta. \tag{2.4}$$

[14] For the general theory of invariant complex line bundles over complex manifolds with discontinuous groups of analytic automorphisms, see Baily [1], [2].

Then the closed real $(1,1)$*-form* $\gamma = (\frac{i}{2\pi})\partial\bar{\partial}\log a(\hat{z})$ *is invariant under* Δ. *Moreover, considered as a closed real* $(1,1)$*-form on* $M = \hat{M}/\Delta$, γ *belongs to the characteristic class* $c_R(F)$ *of the bundle* F.

Proof. It is obvious that γ is invariant under Δ. Now setting

$$a_j(z) = a(\varpi_j^{-1}(z)), \qquad z \in U_j,$$

we infer from (2.3) and (2.4) that

$$a_j(z) = |f_{jk}(z)|^2 a_k(z), \qquad z \in U_j \cap U_k,$$

while we have $\gamma = (\frac{i}{2\pi})\partial\bar{\partial}\log a_j(z)$. Hence, by Theorem 1.1, γ belongs to $c_R(F)$, q.e.d.

In order to construct the complex line bundle Ξ, we apply the above procedure to the product manifold $M = V \times \mathfrak{P}$. Denote by $\hat{V}$ the universal covering of V, by G the covering transformation group of $\hat{V}$ with respect to V, and by ϖ the canonical projection of $\hat{V}$ onto V. Then, since $\mathfrak{P} = \bar{\mathfrak{A}}/\bar{\mathfrak{d}}$, the universal covering $\hat{M}$ of $M = V \times \mathfrak{P}$ is given by $\hat{M} = \hat{V} \times \bar{\mathfrak{A}}$ and the covering transformation group Δ of $\hat{M}$ is given by $\Delta = G \times \bar{\mathfrak{d}}$. Letting $\hat{o}$ be a point on $\hat{V}$ fixed once and for all and denoting by $C(\hat{o}, \hat{z})$ an oriented continuous curve combining $\hat{o}$ with $\hat{z}$, where $\hat{z}$ is a point on V, we associate with each element $g \in G$ the closed continuous curve $C(g) = \varpi C(\hat{o}, g\hat{o})$ on V. As is well known, $g \to C(g)$ induces an isomorphism of G onto the fundamental group of V. Now we set

$$f(\hat{z}, \bar{\alpha}; g, \bar{\delta}) = \exp\{\int_{C(g)} (\bar{\alpha} + \bar{\delta}) + \int_{\hat{o}}^{\hat{z}} \varpi^* \delta\},$$

where $(\hat{z}, \bar{\alpha}) \in \hat{V} \times \bar{\mathfrak{A}} = \hat{M}$, $(g, \bar{\delta}) \in G \times \bar{\mathfrak{d}} = \Delta$ and where $\varpi^*\delta$ is the holomorphic 1-form on $\hat{V}$ induced by $\delta = \bar{\bar{\delta}}$. The non-vanishing function $f(\hat{z}, \bar{\alpha}; g, \bar{\delta})$ is obviously holomorphic in $(\hat{z}, \bar{\alpha})$. Moreover $f(\hat{z}, \bar{\alpha}; g, \bar{\delta})$ satisfies the relation (2.1). In fact, we have

$$\begin{aligned} & f(h\hat{z}, \bar{\alpha} + \bar{\epsilon}; g, \bar{\delta}) \cdot f(\hat{z}, \bar{\alpha}; h, \bar{\epsilon}) \\ &= \exp\{\int_{C(g)} (\bar{\alpha} + \bar{\epsilon} + \bar{\delta}) + \int_{\hat{o}}^{h\hat{z}} \varpi^*\delta + \int_{C(h)} (\bar{\alpha} + \bar{\epsilon}) + \int_{\hat{o}}^{\hat{z}} \varpi^*\epsilon\} \\ &= \exp\{\int_{C(gh)} (\bar{\alpha} + \bar{\delta} + \bar{\epsilon}) + \int_{\hat{o}}^{\hat{z}} \varpi^*(\delta + \epsilon) - \int_{C(h)} \bar{\delta} + \int_{\hat{z}}^{h\hat{z}} \varpi^*\delta\}, \end{aligned}$$

where $h \in G$, $\bar{\epsilon} \in \bar{\mathfrak{d}}$, while, since δ has the form $\delta = \sum m_r \beta_r$, $m_r \in \mathbf{Z}$, we get, using (1.3),

$$\int_{C(h)} \bar{\delta} - \int_{\hat{z}}^{h\hat{z}} \varpi^* \delta = \int_{C(h)} (\bar{\delta} - \delta) = \sum m_r \int_{C(h)} (\bar{\beta}_r - \beta_r) = 2\pi i \sum m_r b_r [C(h)],$$

where $\sum m_r b_r [C(h)]$ is a rational integer. Hence we obtain

$$f(h\hat{z}, \bar{\alpha} + \bar{\epsilon}; g, \bar{\delta}) \cdot f(\hat{z}, \bar{\alpha}; h, \bar{\epsilon}) = f(\hat{z}, \bar{\alpha}; gh, \bar{\delta} + \bar{\epsilon}).$$

Letting $T(g, \delta)$ be the analytic automorphism of $\hat{M} \times \boldsymbol{C} = \hat{V} \times \bar{\mathfrak{A}} \times \boldsymbol{C}$ defined by

$$T(g, \bar{\delta}) : (\hat{z}, \bar{\alpha}, \zeta) \to (g\hat{z}, \bar{\alpha} + \bar{\delta}, f(\hat{z}, \bar{\alpha}; g, \bar{\delta}) \cdot \zeta),$$

we infer therefore that $T(G \times \bar{\mathfrak{d}}) = \{T(g, \bar{\delta}) \mid g \in G, \bar{\delta} \in \bar{\mathfrak{d}}\}$ is a discontinuous group of analytic automorphisms of $\hat{V} \times \bar{\mathfrak{A}} \times \boldsymbol{C}$ and that the factor space

$$\Xi = \hat{V} \times \bar{\mathfrak{A}} \times \boldsymbol{C} / T(G \times \bar{\mathfrak{d}})$$

is a complex line bundle over $V \times \mathfrak{P} = \hat{V} \times \bar{\mathfrak{A}} / G \times \bar{\mathfrak{d}}$.

In order to show that the induced bundle $r_P^*(\Xi)$ over V coincides with P, we take an element $\bar{\alpha} \in \bar{\mathfrak{A}}$ such that $\rho(\bar{\alpha}) = P$. Then we have

$$r_P^*(\Xi) = \hat{V} \times \bar{\alpha} \times \boldsymbol{C} / T(G \times \bar{0}).$$

By the canonical identification $\hat{V} \times \bar{\alpha} \times \boldsymbol{C} = \hat{V} \times \boldsymbol{C}$, each automorphism $T(g, \bar{0})$ is reduced to

$$T(g) : (\hat{z}, \zeta) \to (g\hat{z}, f(\hat{z}, g) \cdot \zeta),$$

where $f(\hat{z}, g) = f(\hat{z}, \bar{\alpha}; g, \bar{0}) = \exp \int_{C(g)} \bar{\alpha}$. Thus the bundle $r_P^*(\Xi)$ is represented in the form $r_P^*(\Xi) = \hat{V} \times \boldsymbol{C} / T(G)$. Now take a finite covering $\{U_j\}$ of V by sufficiently small spherical neighborhoods U_j. To find a system $\{f_{jk}(z)\}$ of transition functions defining the bundle $r_P^*(\Xi)$ with respect to $\{U_j\}$, we apply the formula (2.3) to $r_P^*(\Xi)$. Then, letting ϖ_j have the same meaning as in (2.3), we get

$$f_{jk}(z) = \exp \int_{C(g_{jk})} \bar{\alpha}, \qquad z \in U_j \cap U_k,$$

where $g_{jk} = \varpi_j^{-1} \varpi_k$ on $\varpi_k^{-1}(U_k)$. Take a point $z(j)$ in each U_j and set $\hat{z}(j) = \varpi_j^{-1}(z(j))$. Then we have

$$\int_{C(g_{jk})} \bar{\alpha} = \int_{\varpi_k^{-1}(z)}^{\varpi_j^{-1}(z)} \varpi^* \bar{\alpha} = \{ \int_{\varpi_k^{-1}(z)}^{\hat{z}(k)} + \int_{\hat{z}(k)}^{\hat{z}(j)} + \int_{\hat{z}(j)}^{\varpi_j^{-1}(z)} \} \varpi^* \bar{\alpha}.$$

Setting $\kappa_j = \exp \int_{\hat{o}}^{\hat{z}(j)} \varpi^* \bar{\alpha}$, we obtain therefore

$$f_{jk}(z) = \kappa_j \kappa_k^{-1} \exp \int_{z(j)}^{z(k)} \bar{\alpha}. \tag{2.5}$$

Comparing this with (1.5), we infer that the complex line bundle $r_P^*(\Xi)$ coincides with P (up to an analytic equivalence).

The characteristic class of the complex line bundle Ξ can be easily obtained with the help of the above Lemma 2.1. Letting $\{\bar{\alpha}_1, \bar{\alpha}_2, \cdots, \bar{\alpha}_q\}$ be a base of $\bar{\mathfrak{A}}$ with respect to $\boldsymbol{C}$, we write $\bar{\alpha}$ in the form

$$\bar{\alpha} = t_1 \bar{\alpha}_1 + t_2 \bar{\alpha}_2 + \cdots + t_q \bar{\alpha}_q$$

and use $(t_1, t_2, \cdots, t_q)$ as the "local coordinate" of the point $P = \rho(\bar{\alpha})$. Setting

$$a(\hat{z}, \bar{\alpha}) = \exp \int_{\hat{o}}^{\hat{z}} (\bar{\alpha} + \alpha),$$

we get by a simple computation

$$\partial\bar{\partial} \log a(\hat{z}, \bar{\alpha}) = \sum_{\nu=1}^{q} \{\alpha_\nu \wedge d\bar{t}_\nu - \bar{\alpha}_\nu \wedge dt_\nu\},$$

while, since the positive function $a(\hat{z}, \bar{\alpha})$ satisfies

$$a(g\hat{z}, \bar{\alpha} + \bar{\delta}) = |f(\hat{z}, \bar{\alpha}; g, \bar{\delta})|^2 a(\hat{z}, \bar{\alpha}),$$

we infer from the above Lemma 2.1 that $(\frac{i}{2\pi})\partial\bar{\partial} \log a(\hat{z}, \bar{\alpha})$ belongs to $c_R(\Xi)$. Thus, the characteristic class $c_R(\Xi)$ of Ξ is represented by the closed real $(1,1)$-form

$$\xi = (\tfrac{i}{2\pi}) \sum_{\nu=1}^{q} \{\alpha_\nu \wedge d\bar{t}_\nu - \bar{\alpha}_\nu \wedge dt_\nu\}. \tag{2.6}$$

3. Complete continuous systems. Let $\mathfrak{d} = \{D\}$ be a set of effective divisors D on V and let Λ be a non-singular projective variety. A map $\lambda \to D_\lambda$ of Λ onto $\mathfrak{d}$ is called *analytic*[15] if there exists on $V \times \Lambda$ an effective divisor $\mathfrak{D}$ which cuts out on each subvariety $V \times \lambda$ the divisor $D_\lambda \times \lambda$:

$$\mathfrak{D} \cdot (V \times \lambda) = D_\lambda \times \lambda.$$

Clearly the image $\mathfrak{d}$ of Λ by an analytic map is an irreducible alegbraic system, i.e., a *continuous system* in the sense of Italian geometers. Now, if there exists a one-to-one analytic map $\lambda \to D_\lambda$ of Λ onto $\mathfrak{d}$, then we say that the algebraic system $\mathfrak{d}$ is *parametrized*[16] by Λ and moreover we call

[15] Chow [4], p. 897.
[16] Weil [18], p. 881.

Λ the parameter variety of $\mathscr{S}$. We note that the parameter variety Λ of $\mathscr{S}$ is determined uniquely up to a bi-regular equivalence. In fact, given another non-singular projective variety Λ' and a one-to-one analytic map $\lambda' \to D_{\lambda'}$ of Λ' onto $\mathscr{S}$, we observe the one-to-one correspondence $\lambda \to \lambda'$ between Λ and Λ' defined by $D_\lambda = D_{\lambda'}$. As was shown by A. Weil,[17] this correspondence $\lambda \to \lambda'$ is *analytic* in the sense that the graph of the mapping $\lambda \to \lambda'$ is an analytic subvariety of the product variety $\Lambda \times \Lambda'$, while *any one-to-one analytic correspondence between two non-singular varieties is a bi-regular mapping.*[18] Thus, we infer that Λ' is bi-regularly equivalent to[19] Λ.

Given a divisor D_0 on V, such that $|D_0|$ is not empty, we consider the set $\mathscr{C} = \{D \mid D \sim D_0, D > 0\}$ consisting of all effective divisors $D \sim D_0$ on V, where $\sim$ denotes the homology with coefficients in the integers $\mathbf{Z}$. In this Section we shall prove that, in case the characteristic class $c([D_0] - K)$ of $[D_0] - K$ is positive, K being the canonical bundle over V, the set $\mathscr{C}$ is an algebraic system parametrized by a non-singular projective variety.

We set $E = [D_0]$. Since the characteristic class $c([D])$ of $[D]$ is equal to the dual of the homology class of the $(2n-2)$-cycle D on V, the homology $D \sim D_0$ holds if and only if $[D] - E \in \mathfrak{P}$, while, setting $[D] - E = P$, D is effective if and only if D belongs to $|E + P|$. We infer therefore that

$$\mathscr{C} = \bigcup_{P \in \mathfrak{P}} |E + P|. \tag{3.1}$$

Since $\mathfrak{P}$ is a non-singular algebraic variety imbedded in a projective space, there exists a complex line bundle $\mathfrak{E}_1$ over $\mathfrak{P}$ such that $c_R(\mathfrak{E}_1)$ contains a closed positive $(1,1)$-form ω. Let $\mathfrak{E} = m\mathfrak{E}_1$, where m is a sufficiently large positive integer. $\mathfrak{E}$ and E induce complex line bundles over $V \times \mathfrak{P}$ in an obvious manner which will be denoted by the same symbols $\mathfrak{E}$ and E, respectively. Letting Ξ be the complex line bundle over $V \times \mathfrak{P}$ defined in Section 2, we form the sum

$$F = E + \Xi + \mathfrak{E}, \qquad \text{on } V \times \mathfrak{P}. \tag{3.2}$$

---

[17] Weil [18], p. 883.

[18] For the foundations of algebroid geometry, see Chow [6].

[19] This can be verified also in the following manner: Let $x(D_\lambda)$ be the Chow coordinate of the divisor D_λ. $x(D_\lambda)$ may be taken as a point in a projective space $\mathfrak{X}$ and $\lambda \to x(D_\lambda)$ is a one-to-one mapping of Λ into $\mathfrak{X}$. Moreover the mapping $\lambda \to x(D_\lambda)$ is regular (see Chow [4], p. 897). It follows that the mapping $\lambda \to \lambda'$ is regular except possibly a bunch of proper subvarieties of Λ, while $\lambda \to \lambda'$ is obviously continuous. Hence, by a theorem of Bochner and Martin, the mapping $\lambda \to \lambda'$ is everywhere regular on Λ (see Bochner and Martin [3], p. 173).

Then the bundle $r_P^*(F)$ over V induced by the mapping $r_P: z \to (z, P)$ of V into $V \times \mathfrak{P}$ has the form

$$r_P^*(F) = E + P, \tag{3.3}$$

since $r_P^*(\Xi)$ is equal to P. It follows that r_P induces a homomorphism r_P^* of $\Gamma(F)$ into $\Gamma(E+P)$ in a canonical manner.

LEMMA 3.1. *The homomorphism r_P^* maps $\Gamma(F)$ onto $\Gamma(E+P)$:*

$$r_P^*\Gamma(F) = \Gamma(E+P). \tag{3.4}$$

Proof. We form the quadratic transform $\tilde{\mathfrak{P}} = \sigma_P(\mathfrak{P})$ of $\mathfrak{P}$, where σ_P denotes the *quadratic transformation*[20] with the center P. We note that $S = \sigma_P(P)$ is (bi-regularly equivalent to) a projective space of dimension $q-1$, q being the dimension of $\mathfrak{P}$, and that σ_P^{-1} is a regular map of $\tilde{\mathfrak{P}}$ onto $\mathfrak{P}$ which is bi-regular between $\tilde{\mathfrak{P}} - S$ and $\mathfrak{P} - P$. We denote by N_P the regular map of $V \times \tilde{\mathfrak{P}}$ onto $V \times \mathfrak{P}$ induced by σ_P^{-1} in a canonical manner. Obviously N_P^{-1} is the "monoidal transformation" defined by

$$N_P^{-1}: (z, Q) \to (z, \sigma_P(Q)),$$

where Q denotes a "variable" point on $\mathfrak{P}$. Now let i be the inclusion map of $V \times S$ into $V \times \tilde{\mathfrak{P}}$ and let Π be the canonical projection of $V \times S$ onto V. Then we have the commutative diagram

$$\begin{array}{ccc} V \times \mathfrak{P} & \xleftarrow{r_P} & V \\ \uparrow{\scriptstyle N_P} & & \uparrow{\scriptstyle \Pi} \\ V \times \tilde{\mathfrak{P}} & \xleftarrow{i} & V \times S. \end{array} \tag{3.5}$$

Let $\tilde{F} = N_P^*(F)$ be the induced bundle over $V \times \tilde{\mathfrak{P}}$ and let $\tilde{F}_{V \times S} = i^*(\tilde{F})$ be the restriction of $\tilde{F}$ to $V \times S$. Then, since $r_P^*(F) = E + P$ by (3.3), we infer from (3.5) that

$$\Pi^*(E+P) = \tilde{F}_{V \times S}. \tag{3.6}$$

Moreover we obtain the commutative diagram

$$\begin{array}{ccc} \Gamma(F) & \xrightarrow{r_P^*} & \Gamma(E+P) \\ \downarrow{\scriptstyle N_P^*} & & \downarrow{\scriptstyle \Pi^*} \\ \Gamma(\tilde{F}) & \xrightarrow{i^*} & \Gamma(\tilde{F}_{V \times S}), \end{array} \tag{3.7}$$

[20] For elementary properties of quadratic transformations, see Kodaira [11], § 2.

where N_P^*, Π^*, i^* denote the homomorphisms induced by N_P, Π, i, respectively. In this diagram (3.7), N_P^* and Π^* are both *isomorphisms onto*. In fact, it is obvious by (3.6) that Π^* maps $\Gamma(E+P)$ isomorphically onto $\Gamma(\tilde{F}_{V\times S})$. N_P^* is clearly an isomorphism of $\Gamma(F)$ *into* $\Gamma(\tilde{F})$. Take an arbitrary section $\psi \in \Gamma(\tilde{F})$. Then, since N_P is a biregular map of $V\times\tilde{\mathfrak{P}} - V\times S$ onto $V\times\mathfrak{P} - V\times P$, there exists a holomorphic section ϕ' of F over $V\times\mathfrak{P} - V\times P$ such that $N_P^*\phi' = \psi$ on $V\times\tilde{\mathfrak{P}} - V\times S$. By virtue of a theorem of Hartogs, ϕ' can be extended to $\phi \in \Gamma(F)$. This extension ϕ satisfies obviously $N_P^*\phi = \psi$ on $V\times\tilde{\mathfrak{P}}$. Thus we see that N_P^* maps $\Gamma(F)$ isomorhpically *onto* $\Gamma(\tilde{F})$. To prove our lemma, it is therefore sufficient to show that

$$i^*\Gamma(\tilde{F}) = \Gamma(\tilde{F}_{V\times S}). \tag{3.8}$$

We observe the exact sequence

$$0 \to \Omega(\tilde{F} - [V\times S]) \to \Omega(\tilde{F}) \xrightarrow{r_{V\times S}} \Omega(\tilde{F}_{V\times S}) \to 0.$$

The exact cohomology sequence corresponding to this can be written in the form

$$\cdots \to \Gamma(\tilde{F}) \xrightarrow{i^*} \Gamma(\tilde{F}_{V\times S}) \to H^1(V\times\tilde{\mathfrak{P}}, \Omega(\tilde{F} - [V\times S])) \to \cdots$$

since the homomorphism $r^*_{V\times S}\colon \Gamma(\tilde{F}) \to \Gamma(\tilde{F}_{V\times S})$ coincides with i^*. In order to show (3.8), it is therefore sufficient to prove that

$$H^1(V\times\tilde{\mathfrak{P}}, \Omega(\tilde{F} - [V\times S])) = 0. \tag{3.9}$$

For simplicity's sake we use the following abbreviations: Let M, N be complex manifolds. Given a complex line bundle B [or a form γ] on M, we denote by the same symbol B [or γ] simultaneously the bundle [or the form] on the product space $M\times N$ induced by B [or γ] in an obvious manner. For example we mean by the bundle K over $V\times\mathfrak{P}$ the bundle over $V\times\mathfrak{P}$ induced by the canonical bundle K over V. Now we prove (3.9) with the help of Theorem 1.2. Since the canonical bundle over $\mathfrak{P}$ is trivial, the canonical bundle over $\tilde{\mathfrak{P}}$ is equal[21] to $(q-1)[S]$. Hence the *canonical bundle* $\mathfrak{K}$ *over* $V\times\tilde{\mathfrak{P}}$ is given by $\mathfrak{K} = K + (q-1)[V\times S]$. We have therefore

$$\tilde{F} - [V\times S] - \mathfrak{K} = E - K + \tilde{\Xi} + \tilde{\mathfrak{E}} - q[V\times S], \tag{3.10}$$

where $\tilde{\Xi} = N_P^*(\Xi)$, $\tilde{\mathfrak{E}} = N_P^*(\mathfrak{E})$. Setting $m = m' + m''$, we decompose $\mathfrak{E}$ in the right hand side of (3.10) into two parts:

$$\mathfrak{E} = \mathfrak{E}' + \mathfrak{E}'', \qquad \text{where } \mathfrak{E}' = m'\mathfrak{E}_1, \mathfrak{E}'' = m''\mathfrak{E}_1.$$

[21] Kodaira [11], p. 31.

By hypothesis $c_R(E-K)$ contains a closed *positive* $(1,1)$-form γ on V. We have therefore

$$c_R(E-K+\Xi+\mathfrak{E}') \ni \gamma+\xi+m'\omega, \qquad \text{on } V\times\mathfrak{P},$$

where ξ is the form defined by (2.6) belonging to $c_R(\Xi)$. Clearly the restriction $\xi_{V\times Q}$ of ξ to $V\times Q$ vanishes for each point $Q\in\mathfrak{P}$, while ω is positive on $\mathfrak{P}$. Hence we get

$$\gamma+\xi+m'\omega>0. \qquad \text{on } V\times\mathfrak{P},$$

provided that m' is sufficiently large. On the other hand, $c_R(\tilde{\mathfrak{E}}''-q[S])$ contains a closed positive $(1,1)$-form[22] ρ'' on $\tilde{\mathfrak{P}}$, provided that m'' is sufficiently large, where $\tilde{\mathfrak{E}}''=\sigma_P(\mathfrak{E}'')$ is the bundle over $\tilde{\mathfrak{P}}$ induced by $\mathfrak{E}''=m''\mathfrak{E}_1$. Setting $\rho=\gamma+\xi+m'\omega$ on $V\times\mathfrak{P}$, we therefore infer from (3.10) that

$$c_R(\tilde{F}-[V\times S]-\mathfrak{K}) \ni \tilde{\rho}+\rho'', \qquad \text{on } V\times\tilde{\mathfrak{P}},$$

where $\tilde{\rho}=N_P^*(\rho)$. To show that $\tilde{\rho}+\rho''$ is positive on $V\times\tilde{\mathfrak{P}}$, we observe that

$$\Psi\colon (z,\tilde{Q})\to(z,Q,\tilde{Q}), \qquad Q=\sigma_P^{-1}(\tilde{Q}),$$

is a bi-regular map of $V\times\tilde{\mathfrak{P}}$ into $V\times\mathfrak{P}\times\tilde{\mathfrak{P}}$, where $(z,\tilde{Q})$ denotes a "variable" point on $V\times\tilde{\mathfrak{P}}$. Since ρ or ρ'' is positive on $V\times\mathfrak{P}$ or $\tilde{\mathfrak{P}}$, the sum $\rho+\rho''$ is positive on $V\times\mathfrak{P}\times\tilde{\mathfrak{P}}$, while $\tilde{\rho}+\rho''=\Psi^*(\rho+\rho'')$. Consequently $\tilde{\rho}+\rho''$ is positive on $V\times\tilde{\mathfrak{P}}$. Thus we see that

$$c_R(\tilde{F}-[V\times S]-\mathfrak{K})>0, \qquad \text{on } V\times\tilde{\mathfrak{P}}.$$

and therefore, by using Theorem 1.2, we obtain (3.9), q.e.d.

LEMMA 3.2.[23] *Let B be an arbitrary complex line bundle over V and let Q be a member of $\mathfrak{P}$. Then we have*

$$\limsup \dim\Gamma(B+P) \leqq \dim\Gamma(B+Q) \quad \textit{as } P\to Q, \tag{3.11}$$

where P is a "variable" member of $\mathfrak{P}$.

Proof. Take a sufficiently fine covering $\{U_j\}$ of V. Then the complex line bundle P is defined by the system $\{\kappa_{jk}\}$ of the constants

$$\kappa_{jk}=\exp\int_{z(j)}^{z(k)}\tilde{\alpha},$$

[22] Kodaira [11], p. 31, Lemma 1.

[23] This lemma is reduced to a special case of a theorem of Kodaira and Spencer concerning semi-continuity of the dimensions of cohomologies. See Kodaira and Spencer [14], p. 15. We give here a simplified version of their proof for this special case.

where $\bar{\alpha}$ is an element of $\bar{\mathfrak{A}}$ such that $\rho(\bar{\alpha}) = P$ (see (1.5)). The system $\{\bar{\kappa}_{jk}\}$ defines an analytically trivial bundle, since

$$\bar{\kappa}_{jk} = f_j(z)/f_k(z), \qquad \text{where } f_j(z) = \exp \int_{z(j)}^{z} \alpha.$$

Hence we may replace $\{\kappa_{jk}\}$ by the system $\{\theta_{jk}\}$ of constants $\theta_{jk} = \kappa_{jk}/\bar{\kappa}_{jk}$ with $|\theta_{jk}| = 1$. Now let $\{b_{jk}(z)\}$ be the system of transition functions defining the bundle B and let $\{a_j(z)\}$ be the system of functions $a_j(z) > 0$ of class C^∞ on U_j such that $a_j(z) = |b_{jk}(z)|^2 a_k(z)$ in $U_j \cap U_k$. For any section $\phi: z \to \phi(z)$ belonging to $\Gamma(B+P)$, the fibre coordinates $\phi_j(z)$ of $\phi(z)$ satisfy

$$\phi_j(z) = b_{jk}(z)\theta_{jk} \cdot \phi_k(z), \qquad \text{in } U_j \cap U_k,$$

and therefore $a_j(z)^{-1} |\phi_j(z)|^2 = a_k(z)^{-1} |\phi_k(z)|^2$. Hence we may define the inner product

$$(\phi', \phi) = \int_V a_j(z)^{-1} \phi_j'(z) \bar{\phi}_j(z)\, dV, \qquad \text{for } \phi', \phi \in \Gamma(B+P),$$

where dV denotes the invariant volume element on V. Now let

$$l = \limsup \dim \Gamma(B+P) \text{ as } P \to Q.$$

Then there exists a sequence $P_1, P_2, \cdots, P_m, \cdots$ with $\lim P_m = Q$ such that $\dim \Gamma(B+P_m) = l$ for $m = 1, 2, \cdots$. We take $\bar{\alpha}_m, \bar{\alpha}_\infty \in \bar{\mathfrak{A}}$ with $\rho(\bar{\alpha}_m) = P_m$, $\rho(\bar{\alpha}_\infty) = Q$ in such a way that $\bar{\alpha}_m \to \bar{\alpha}_\infty$ for $m \to \infty$, and denote by $\theta_{jk}^{(m)}$ the θ_{jk} corresponding to $\bar{\alpha}_m$. Moreover we form a base

$$\{\phi_1^{(m)}, \cdots, \phi_\nu^{(m)}, \cdots, \phi_l^{(m)}\}$$

of each $\Gamma(B+P_m)$ satisfying $(\phi_\lambda^{(m)}, \phi_\nu^{(m)}) = \delta_{\lambda\nu}$. We have

$$\int_{U_j} a_j(z)^{-1} |\phi_{\nu j}^{(m)}(z)|^2\, dV \leqq (\phi_\nu^{(m)}, \phi_\nu^{(m)}) = 1.$$

There exists therefore a subsequence [24] of $\{\phi_{\nu j}^{(1)}, \phi_{\nu j}^{(2)}, \cdots, \phi_{\nu j}^{(m)}, \cdots\}$ which converges uniformly on each compact subset of U_j. Consequently, by a suitable choice of the sequence $\{P_m\}$, we may assume that, as $m \to \infty$, the limit $\phi_{\nu j}(z) = \lim \phi_{\nu j}^{(m)}(z)$ exists and represents a holomorphic function on U_j. Since

$$\phi_{\nu j}^{(m)}(z) = b_{jk}(z)\theta_{jk}^{(m)} \cdot \phi_{\nu k}^{(m)}(z), \qquad \text{in } U_j \cap U_k,$$

we have

$$\phi_{\nu j}(z) = b_{jk}(z)\theta_{jk}^{(\infty)} \cdot \phi_{\nu k}(z), \qquad \text{in } U_j \cap U_k,$$

[24] Bochner and Martin [3], pp. 116-118.

where $\theta_{jk}^{(\infty)} = \lim \theta_{jk}^{(m)}$ is the θ_{jk} corresponding to $\tilde{\alpha}_\infty$. This shows that $\phi_{\nu j}(z)$ represents the fibre coordinate over U_j of a section $\phi_\nu \in \Gamma(B+Q)$. Moreover we have

$$(\phi_\lambda, \phi_\nu) = \lim_{m\to\infty} (\phi_\lambda^{(m)}, \phi_\nu^{(m)}) = \delta_{\lambda\nu}.$$

Thus we infer that $\Gamma(B+Q)$ contains l linearly independent elements $\phi_1, \cdots, \phi_\nu, \cdots, \phi_l$ and therefore we obtain (3.11).

LEMMA 3.3. *The dimension of* $\Gamma(E+P)$ *is independent of* $P \in \mathfrak{P}$, *provided that* $c(E-K)$ *is positive on* V.

Proof. Take an arbitrary point $Q \in \mathfrak{P}$ and set $\dim \Gamma(E+Q) = d+1$. We readily infer from the above Lemma 3.1 that

$$\liminf \dim \Gamma(E+P) \geqq d+1 \text{ as } P \to Q. \tag{3.12}$$

In fact, since every element ϕ of $\Gamma(E+Q)$ can be written in the form $\phi = r_Q{}^*\psi$, $\psi \in \Gamma(F)$, there exist $d+1$ elements $\psi_0, \psi_1, \cdots, \psi_d$ of $\Gamma(F)$ such that $r_Q{}^*\psi_0, r_Q{}^*\psi_1, \cdots, r_Q{}^*\psi_d$ are linearly independent. It follows that $d+1$ elements $r_P{}^*\psi_0, r_P{}^*\psi_1, \cdots, r_P{}^*\psi_d$ of $\Gamma(E+P)$ are linearly independent if P lies in a sufficiently small neighborhood of Q. Hence we obtain the inequality (3.12). Now, comparing (3.12) with (3.11), we infer immediately that the integer-valued function $\dim \Gamma(E+P)$ in P reduces to a constant, q.e.d.

By means of the above Lemmas 3.1 and 3.3 we define an analytic bundle structure on the set $\mathfrak{B} = \bigcup_{P\in\mathfrak{P}} \Gamma(E+P)$ in the following manner: First we define the projection $\pi\colon \mathfrak{B} \to \mathfrak{P}$ by $\pi(\phi) = P$ for $\phi \in \Gamma(E+P)$. Given a point $Q \in \mathfrak{P}$, we take $d+1$ elements $\psi_0, \psi_1, \cdots, \psi_d \in \Gamma(F)$ such that $r_Q{}^*\psi_0, r_Q{}^*\psi_1, \cdots, r_Q{}^*\psi_d$ form a base of $\Gamma(E+Q)$, where $d+1 = \dim \Gamma(E+Q)$. This is possible because of Lemma 3.1. Obviously $r_P{}^*\psi_0, \cdots, r_P{}^*\psi_d \in \Gamma(E+P)$ are linearly independent if P lies in a sufficiently small neighborhood $\mathfrak{U}(Q)$ of Q, while, by Lemma 3.3, $\dim \Gamma(E+P) = d+1$. Hence $r_P{}^*\psi_0, \cdots, r_P{}^*\psi_d$ form a base of $\Gamma(E+P)$ for each $P \in \mathfrak{U}(Q)$. An arbitrary element $\phi \in \pi^{-1}(\mathfrak{U}(Q))$ can therefore be written in the form

$$\phi = \sum_{\nu=0}^{d} \zeta_\nu \cdot r_P{}^*\psi_\nu, \qquad \zeta_\nu \in \boldsymbol{C}. \tag{3.13}$$

In case $\phi \in \pi^{-1}(\mathfrak{U}(Q) \cap \mathfrak{U}(Q'))$, we have two representations

$$\phi = \sum_{\nu=0}^{d} \zeta_\nu \cdot r_P{}^*\psi_\nu = \sum_{\nu=0}^{d} \zeta_\nu' \cdot r_P{}^*\psi_\nu',$$

where $\psi_0', \cdots, \psi_d'$ are elements of $\Gamma(F)$ such that $r_{Q'}{}^*\psi_0', \cdots, r_{Q'}{}^*\psi_d'$ form a base of $\Gamma(E+Q')$. Obviously we have

$$r_P{}^*\psi_\nu = \sum_{\mu=0}^{d} \Theta_{\mu\nu}(P) \cdot r_P\psi_\mu' \tag{3.14}$$

and therefore

$$\zeta_\mu' = \sum_{\nu=0}^{d} \Theta_{\mu\nu}(P)\zeta_\nu, \tag{3.15}$$

where the coefficients $\Theta_{\mu\nu}(P)$ are holomorphic functions of $P \in \mathfrak{U}(Q) \cap \mathfrak{U}(Q')$ and $\det(\Theta_{\mu\nu}(P)) \neq 0$. Now we define the bundle structure on $\mathfrak{B}$ by assigning, for each $Q \in \mathfrak{P}$, the product structure on $\pi^{-1}(\mathfrak{U}(Q))$ induced by the one-to-one correspondence

$$(P, \zeta_0, \zeta_1, \cdots, \zeta_d) \to \phi = \sum_{\nu=0}^{d} \zeta_\nu \cdot r_P{}^*\psi_\nu$$

between $\mathfrak{U}(Q) \times \boldsymbol{C}_{d+1}$ and $\pi^{-1}(\mathfrak{U}(Q))$. It follows from (3.13) and (3.15) that the bundle $\mathfrak{B}$ thus defined is an analytic fibre bundle over $\mathfrak{P}$ whose fibre is the complex vector space $\boldsymbol{C}_{d+1}$ and whose structure group is the general linear group $GL(d+1, \boldsymbol{C})$.

Now replacing each fibre $\boldsymbol{C}_{d+1}$ of $\mathfrak{B}$ by the corresponding projective space $\mathfrak{S}_d$, we derive from $\mathfrak{B}$ a projective bundle Λ over $\mathfrak{P}$. More precisely, we first from the product space $\mathfrak{U}(Q) \times \mathfrak{S}_d$ for each point $Q \in \mathfrak{P}$, and then we derive the bundle Λ from the collection of the product spaces $\mathfrak{U}(Q) \times \mathfrak{S}_d$, $Q \in \mathfrak{P}$, by identifying $(P, \zeta) \in \mathfrak{U}(Q) \times \mathfrak{S}_d$ with $(P, \zeta') \in \mathfrak{U}(Q') \times \mathfrak{S}_d$ if and only if

$$\kappa \cdot \zeta_\mu' = \sum_{\nu=0}^{d} \Theta_{\mu\nu}(P)\zeta_\nu, \qquad \kappa \neq 0, \tag{3.16}$$

where $(\zeta_0, \cdots, \zeta_d)$ and $(\zeta_0', \cdots, \zeta_d')$ denote the homogeneous coordinates of ζ and ζ', respectively. The projective bundle Λ over $\mathfrak{P}$ is a non-singular projective variety,[25] since the base space $\mathfrak{P}$ is a non-singular projective variety. Let $\mathfrak{B}^*$ be the subspace of $\mathfrak{B}$ obtained by deleting the origin $(0, \cdots, 0)$ of each fibre $\boldsymbol{C}_{d+1}$ of $\mathfrak{B}$. Then we have the canonical mapping

$$\phi = (P, \zeta_0, \cdots, \zeta_d) \to \lambda(\phi) = (P, \zeta)$$

of $\mathfrak{B}^*$ onto Λ, where ζ is the point in $\mathfrak{S}_d$ with the homogeneous coordinate $(\zeta_0, \zeta_1, \cdots, \zeta_d)$. Since the divisor $D = (\phi)$ of $\phi = \sum \zeta_\nu r_P{}^*\psi_\nu$ is determined uniquely by $\lambda(\phi) = (P, \zeta)$, we set

$$D_\lambda = (\phi), \qquad \text{for } \lambda = \lambda(\phi). \tag{3.17}$$

[25] Kodaira [11], p. 42, Theorem 8.

In view of (3.1), the mapping $\phi \to D = (\phi)$ maps $\mathfrak{B}^*$ onto $\mathfrak{C} = \bigcup_{P \in \mathfrak{P}} |E + P|$, and moreover we have $(\phi') = (\phi)$ if and only if $\lambda(\phi') = \lambda(\phi)$. Consequently $\lambda \to D_\lambda$ gives a one-to-one correspondence between Λ and $\mathfrak{C}$. Moreover the correspondence $\lambda \to D_\lambda$ is *algebraic*, i.e., there exists an effective divisor $\mathfrak{D}$ on the product variety $V \times \Lambda$ such that

$$D_\lambda \times \lambda = (V \times \lambda) \cdot \mathfrak{D}, \qquad \text{for all } \lambda \in \Lambda. \tag{3.18}$$

This can be verified as follows: For any section $\psi : (z, P) \to \psi(z, P)$ belonging to $\Gamma(F)$, we denote by $\psi_{jQ}(z, P)$ the fibre coordinate of $\psi(z, P)$ over $U_j \times \mathfrak{U}(Q)$. Moreover we denote by $\{f_{jQkQ'}(z, P)\}$ the system of transition functions defining the bundle F with respect to the covering $\{U_j \times \mathfrak{U}(Q)\}$ of $V \times \mathfrak{P}$. We note that, for $P \in \mathfrak{U}(Q)$, the induced bundle $E + P = r_P{}^*F$ over V is defined by the system $\{f_{jQkQ}(z, P)\}$ with respect to the covering $\{U_j\}$ and that the mapping $\psi \to r_P{}^*\psi$ is simply given by

$$(r_P{}^*\psi)_j(z) = \psi_{jQ}(z, P). \tag{3.19}$$

Therefore it follows from (3.14) that

$$\psi_{\nu jQ}(z, P) = \sum_\mu \Theta_{\mu\nu}(P) \psi'_{\mu jQ}(z, P). \tag{3.20}$$

Now let $\mathfrak{D}_\mu$ be the open subset $\{\zeta \mid \zeta_\mu \neq 0\}$ of $\mathfrak{S}_d$. Then

$$R_{jQ\mu}(z, \lambda) = \sum_{\nu=0}^{d} (\zeta_\nu / \zeta_\mu) \psi_{\nu jQ}(z, P) \qquad \text{where } \lambda = (P, \zeta), \tag{3.21}$$

is a holomorphic function of (z, λ) defined on the open set

$$\mathfrak{U}_{jQ\mu} = U_j \times \mathfrak{U}(Q) \times \mathfrak{D}_\mu \subset V \times \Lambda.$$

Moreover, since, by (3.20),

$$\psi_{\nu jQ}(z, P) = f_{jQkQ'}(z, P) \sum_\mu \Theta_{\mu\nu}(P) \psi'_{\mu kQ'}(z, P)$$

for $z \in U_j \cap U_k$, $P \in \mathfrak{A}(Q) \cap \mathfrak{A}(Q')$, we infer from (3.16) that

$$R_{jQ\mu}(z, \lambda) / R_{kQ'\nu}(z, \lambda) = f_{jQkQ'}(z, P) \sum_\tau \Theta_{\nu\tau}(P) \cdot (\zeta_\tau / \zeta_\mu), \tag{3.22}$$

in $\mathfrak{U}_{jQ\mu} \cap \mathfrak{U}_{kQ'\nu}$.

The right hand side of (3.22) is obviously a non-vanishing holomorphic function on $\mathfrak{U}_{jQ\mu} \cap \mathfrak{U}_{kQ'\nu}$. Hence we can define an effective divisor $\mathfrak{D}$ on $V \times \Lambda$ by

$$\mathfrak{D} = (R_{jQ\mu}), \qquad \text{on each } \mathfrak{U}_{jQ\mu}.$$

Clearly the fibre coordinate $\phi_j(z)$ of $\phi = \sum_\nu \zeta_\nu \cdot r_P{}^*\psi_\nu$ is written in the form

$$\phi_j(z) = \zeta_\mu \cdot R_{jQ\mu}(z, \lambda), \qquad \text{for } \lambda = (P, \zeta) \in \mathfrak{U}(Q) \times \mathfrak{D}_\mu.$$

Hence the divisor $D_\lambda = (\phi)$ is given by $D_\lambda = (R_{jQ\mu}(\ , \lambda))$, on U_j. This proves (3.18). Thus we obtain the following

THEOREM 3.1. *Let D_0 be a divisor on V such that $|D_0|$ is not empty and let $\mathfrak{k}$ be the set consisting of all effective divisors $D \sim D_0$ on V, where $\sim$ denotes the homology with coefficients in the integers. Assume that*

$$c([D_0] - K) > 0, \qquad \text{on } V, \tag{c}$$

where K is the canonical bundle over V. Then the set $\mathfrak{k}$ forms an irreducible algebraic system of effective divisors parametrized by a non-singular projective vairety Λ in the sense that there exists a one-to-one analytic map $\lambda \to D_\lambda$ of Λ onto $\mathfrak{k}$. The parameter variety Λ of $\mathfrak{k}$ is an analytic fibre bundle over the Picard variety $\mathfrak{P}$ attached to V whose fibre is a projective space $\mathfrak{S}_d$ of dimension $d = \dim |D_0|$ and whose structure group is the projective transformation group acting on $\mathfrak{S}_d$. Moreover the canonical projection p of Λ onto $\mathfrak{P}$ is given by

$$p: \lambda \to p(\lambda) = [D_\lambda - D_0].$$

We note that the condition (c) in the above theorem is equivalent to

(c′) $\quad |m(D_0 - K)|$ is ample for sufficiently large integer m.

The complex line bundle $[\mathfrak{D}]$ over $V \times \Lambda$ associated with the divisor $\mathfrak{D}$ can be interpreted in the following manner: The subspace $\mathfrak{B}^*$ of $\mathfrak{B}$ is obviously a principal bundle over Λ whose structure group is $\boldsymbol{C}^*$. By replacing each fibre $\boldsymbol{C}^*$ of $\mathfrak{B}^*$ by $\boldsymbol{C}$, we obtain from $\mathfrak{B}^*$ a complex line bundle B over Λ. B is defined with respect to the covering $\{\mathfrak{U}(Q) \times \mathfrak{D}_\mu\}$ of Λ by the system $\{b_{Q\mu Q'\nu}(\lambda)\}$ of the functions

$$b_{Q\mu Q'\nu}(\lambda) = \zeta_\mu / \sum_\tau \Theta_{\nu\tau}(P)\zeta_\tau, \qquad \lambda = (P, \zeta).$$

Hence it follows from (3.22) that

$$[\mathfrak{D}] = F - B, \tag{3.23}$$

where F in the right hand side denotes the complex line bundle over $V \times \Lambda$ induced by the bundle F over $V \times \mathfrak{P}$ in a canonical manner.

By a *continuous system* of divisors on V we shall mean an *irreducible* algebraic system of effective divisors on V. We say that two effective divisors

D and D' on V are *algebraically equivalent in the restricted sense*[26] and write $D \,|||\, D'$ if D and D' belong to the same continuous system. Moreover, for two (not necessarily effective) divisors D, D', we say that D and D' are *algebraically equivalent* if there exists a divisor X on V such that $D+X \,|||\, D'+X$. It is not difficult to show that $D \,|||\, D'$ *implies* $D \sim D'$. Conversely, *if* $D \sim D'$, *then the strict algebraic equivalence* $D+X \,|||\, D'+X$ *holds for any divisor* X *with sufficiently positive characteristic class* $c(X)$, as one readily infers from Theorem 3.1. Thus we obtain a well known theorem[27] to the effect that *the algebraic equivalence for divisors on a non-singular projective variety coincides with the homology equivalence.* A continuous system is said to be *complete* if it is not contained in a larger continuous system. A given effective divisor does not always determine uniquely the complete continuous system containing it. The continuous system $\mathfrak{C}$ introduced in Theorem 3.1 is obviously complete. Moreover, $\mathfrak{C}$ is the only complete continuous system containing the given divisor D_0.

4. Characteristic linear systems of complete continuous systems. Let C be an effective divisor on V and let $\mathfrak{M}$ be the set of all effective divisors $X \sim C$ on V. A "canonical parametrization" of this set $\mathfrak{M}$ can be introduced in the following manner[28]: Take a general hypersurface section L of V (in one of its ambient projective spaces) such that

$$c([C+L]-K) > 0. \tag{4.1}$$

Then, by Theorem 3.1, the set $\mathfrak{C}$ of all effective divisors $D \sim C+L$ on V forms a complete continuous system parametrized by a projective bundle Λ over the Picard variety $\mathfrak{P}$ attached to V:

$$\mathfrak{C} = \{D_\lambda \mid \lambda \in \Lambda\},$$

where the canonical projection p of Λ onto $\mathfrak{P}$ is given by

$$p\colon \lambda \to p(\lambda) = [D_\lambda - D_0], \qquad D_0 = C+L.$$

Now, letting $M = \{\lambda \mid D_\lambda > L\}$, we set $C_\lambda = D_\lambda - L$ for $\lambda \in M$. Then we infer readily that

$$\mathfrak{M} = \{C_\lambda \mid \lambda \in M\}, \tag{4.2}$$

while M is a (possibly reducible) algebraic subvariety of Λ. Thus the one-

[26] Zariski [19], p. 79.
[27] Lefschetz [16], Chap. IV; see also Igusa [7], p. 16.
[28] Weil [18], p. 887.

to-one mapping $\lambda \to C_\lambda$ of M onto $\mathfrak{M}$ gives a parametrization of $\mathfrak{M}$, which we call the *canonical parametrization* of $\mathfrak{M}$. In what follows we denote a point on M by μ instead of λ and write $\mu = \mu(X)$ if $X = C_\mu$.

LEMMA 4.1. *If C_o, $o \in M$, satisfies the condition*

$$H^1(V, \Omega(C_o)) = 0, \tag{a}$$

then $o = \mu(C_o)$ is a simple point of M. Moreover, for a sufficiently small neighborhood $\mathfrak{U}$ of $p(o)$ on $\mathfrak{P}$, $M \cap p^{-1}(\mathfrak{U})$ is an analytic sub-bundle of $\Lambda \cap p^{-1}(\mathfrak{U})$ in the sense that, for $P \in \mathfrak{U}$, $M \cap p^{-1}(P)$ is a linear sub-variety of the fibre $\mathfrak{S}_d = p^{-1}(P)$ depending holomorphically on P. The dimension h of the fibre $M \cap p^{-1}(P)$ is equal to $\dim |C_o|$.

Proof. Letting $B = [C_o]$ and $E = [D_o]$, $D_o = C_o + L$, we observe the exact sequence

$$0 \to \Omega(B + P) \to \Omega(E + P) \xrightarrow{r_L} \Omega((E + P)_L) \to 0,$$

where $P \in \mathfrak{P}$ and r_L denotes the restriction map to L. From the corresponding exact cohomology sequence

$$0 \to \Gamma(B + P) \to \Gamma(E + P) \to \Gamma((E + P)_L) \to H^1(V, \Omega(B + P)) \to \cdots$$

it follows that

$$0 \leqq \dim \Gamma(B + P) + \dim \Gamma((E + P)_L) - d - 1 \leqq \dim H^1(V, \Omega(B + P)),$$

where $d + 1 = \dim \Gamma(E + P)$. By hypothesis $\dim H^1(V, \Omega(B)) = 0$, while d is independent of P. Hence we get

$$\dim \Gamma(B + P) + \dim \Gamma((E + P)_L) \geqq \dim \Gamma(B) + \dim \Gamma(E_L).$$

On the other hand, $\dim \Gamma(B + P)$ and $\dim \Gamma((E + P)_L)$ are both upper semi-continuous functions in P, as Lemma 3.2 shows. Consequently, letting $h = \dim |C_o|$, we infer that there exists a neighborhood $\mathfrak{U}$ of $0 = p(o)$ on $\mathfrak{P}$ such that

$$\dim |C_o + P| = h, \qquad \text{for } P \in \mathfrak{U}. \tag{4.3}$$

It follows from (3.13) and (3.17) that, for any point

$$\lambda = (P, \zeta) \in p^{-1}(\mathfrak{U}) = \mathfrak{U} \times \mathfrak{S}_d,$$

the divisor D_λ is given by $D_\lambda = (\phi)$, $\phi = \sum_{\nu=0}^{d} \zeta_\nu \psi_\nu(P)$, where $\psi_\nu(P) = r^*{}_P \psi_\nu$. For each point $P \in \mathfrak{U}$, we have therefore

$$M \cap p^{-1}(P) = \{\mu \mid \mu = (P, \zeta), \quad \textstyle\sum \zeta_\nu \psi_\nu(P)_L = 0\},$$

where $\psi_\nu(P)_L$ denotes the restriction of $\psi_\nu(P)$ to L. Since

$$P = p(\mu) = [D_\mu - D_o] = [C_\mu - C_o], \qquad \text{for } \mu \in M,$$

we have $|C_o + P| = \{C_\mu \mid \mu \in M \cap p^{-1}(P)\}$. Combined with (4.3) this shows that $\dim [M \cap p^{-1}(P)] = h$, for $P \in \mathfrak{U}$. Therefore we may assume that $\psi_{h+1}(P)_L, \cdots, \psi_d(P)_L$ are linearly independent and that

$$(4.4) \qquad \psi_\nu(P)_L = \sum_{\tau=h+1}^{d} A_{\nu\tau}(P)\psi_\tau(P)_L, \qquad (\nu = 0, 1, \cdots, h),$$

for $P \in \mathfrak{U}$, provided that $\mathfrak{U}$ is sufficiently small. It follows that the linear equation $\sum_{\nu=0}^{d} \zeta_\nu \psi_\nu(P)_L = 0$ is equivalent to the simultaneous equations

$$\zeta_\tau + \sum_{\nu=0}^{h} A_{\nu\tau}(P)\zeta_\nu = 0, \qquad (\tau = h+1, \cdots, d).$$

Consequently $M \cap p^{-1}(\mathfrak{U})$ is the subset of $p^{-1}(\mathfrak{U}) = \mathfrak{U} \times \mathfrak{S}_o$ consisting of all points of the form

$$(4.5) \qquad \mu = (P, \zeta), \qquad \zeta_\tau = -\sum_{\nu=0}^{h} A_{\nu\tau}(P)\zeta_\nu, \qquad P \in \mathfrak{U},$$

while it is easy to see that the coefficients $A_{\nu\tau}(P)$ determined by (4.4) are holomorphic functions of P. Thus $M \cap p^{-1}(\mathfrak{U})$ is an analytic sub-bundle of $p^{-1}(\mathfrak{U})$, q.e.d.

The canonical parametrization of $\mathfrak{M}$ *is unique,* i.e., it is independent of the choice of the auxiliary hypersurface section L of V. In fact, take any general hypersurface section L^* (in an arbitrary ambient projective space of V) satisfying (4.1) and construct the corresponding parametrization

$$\mathfrak{M} = \{C_{\mu^*} \mid \mu^* \in M^*\}, \qquad M^* \subset \Lambda^*.$$

Then the one-to-one mapping

$$\mu(X) \to \mu^*(X), \qquad X \in \mathfrak{M}$$

of M onto M^* is bi-regular in the sense that there exists a non-singular algebraic variety Σ and bi-regular mappings

$$\Phi\colon \Lambda \to \Sigma, \qquad \Phi^*\colon \Lambda^* \to \Sigma$$

of Λ, Λ^* *into* Σ such that $\Phi(\mu(X)) = \Phi^*(\mu^*(X))$, $X \in \mathfrak{M}$. This fact can be verified as follows: Consider the complete continuous system

$$\mathfrak{L}^{**} = \{D^{**} \mid D^{**} \sim C + L + L^*, D^{**} > 0\}.$$

Since, by hypothesis, $c([C+L+L^*]-K) > c([C+L]-K) > 0$, the system $\mathfrak{B}^{**}$ is parametrized by a non-singular algebraic variety $\mathfrak{Z}$:

$$\mathfrak{B}^{**} = \{D_\sigma^{**} \mid \sigma \in \mathfrak{Z}\}.$$

Letting $\sigma(D^{**}) = \sigma$ for $D^{**} = D_\sigma^{**}$, we define a one-to-one mapping Φ of Λ into $\mathfrak{Z}$ by

$$\Phi\colon \lambda \to \Phi(\lambda) = \sigma(D_\lambda + L^*).$$

Clearly we have $\Phi(\Lambda) = \{\sigma \mid D_\sigma^{**} > L^*\}$. Moreover, for each $\sigma = \Phi(\lambda)$, $D_\lambda = D_\sigma^{**} - L^*$ satisfies $c([D_\lambda]-K) = c([C+L]-K) > 0$, and therefore, by Theorem 1.2, $H^1(V, \Omega(D_\lambda)) = 0$. Hence we infer from the above Lemma 4.1 that $\Phi(\Lambda)$ is a non-singular subvariety of $\mathfrak{Z}$. On the other hand, the one-to-one mapping Φ of Λ onto $\Phi(\Lambda) \subset \mathfrak{Z}$ is analytic.[29] Consequently Φ is a bi-regular mapping[30] of Λ into $\mathfrak{Z}$. Similarly the mapping

$$\Phi^*\colon \lambda^* \to \Phi^*(\lambda^*) = \sigma(D_{\lambda^*}{}^* + L)$$

is a bi-regular mapping of Λ^* into $\mathfrak{Z}$. Moreover we have

$$\Phi(\mu(X)) = \sigma(X + L^* + L) = \Phi^*(\mu^*(X)), \qquad \text{for } X \in \mathfrak{M}.$$

We decompose the parameter variety M into the sum

$$M = M' \cup M'' \cup \cdots \cup M^{(\iota)} \cup \cdots$$

of irreducible components $M', M'', \cdots$ and set $\mathfrak{M}^{(\iota)} = \{C_\mu \mid \mu \in M^{(\iota)}\}$. Obviously we have the corresponding decomposition

$$\mathfrak{M} = \mathfrak{M}' \cup \mathfrak{M}'' \cup \cdots \cup \mathfrak{M}^{(\iota)} \cup \cdots.$$

Each component $\mathfrak{M}^{(\iota)}$ is a *complete continuous system*, i.e., an *irreducible algebraic system parametrized by* $M^{(\iota)}$. By the dimension of the complete continuous system $\mathfrak{M}^{(\iota)}$ we mean the dimension of the parameter variety $M^{(\iota)}$.

Suppose that $C = C_o$ is a non-singular prime divisor belonging to $\mathfrak{M}'$ and that $o = \mu(C)$ is a *simple point* of M'. Then the one-to-one mapping $\mu \to C_\mu$ is analytic in a neighborhood $\mathfrak{V}_o'$ of o on M' in the sense that there exists a divisor $\mathfrak{D}'$ on the open manifold $V \times \mathfrak{V}_o'$ such that

$$\mathfrak{D}' \cdot (V \times \mu) = C_\mu \times \mu, \qquad \mu \in \mathfrak{V}_o'. \tag{4.6}$$

In fact, letting $\{U_j\}$ be a sufficiently fine finite covering of V and letting $\mathfrak{U}_o$ be a neighborhood of o on Λ, we denote by $R_j(z, \lambda)$ the holomorphic function on $U_j \times \mathfrak{U}_o$ defining the divisor $\mathfrak{D}$ which determines the mapping $\lambda \to D_\lambda$

[29] Weil [18], p. 883.

[30] See footnotes 18 and 19.

of Λ onto $\mathfrak{k}$. Moreover, let $w_j(z)$ be the holomorphic function on U_j defining the divisor L. Then, using the fact that L is a non-singular prime divisor, we infer readily that

$$S_j(z,\mu) = R_j(z,\mu)/w_j(z) \tag{4.7}$$

is a holomorphic function on $U_j \times \mathfrak{V}_o'$ and that $b_{jk}(z,\mu) = S_j(z,\mu)/S_k(z,\mu)$ is a non-vanishing holomorphic function on $U_j \times \mathfrak{V}_o' \cap U_k \times \mathfrak{V}_o'$, provided that $\mathfrak{V}_o' \subset \mathfrak{U}_o$. Hence $\mathfrak{D}' = (S_j(z,\mu))$ is a well defined divisor on $V \times \mathfrak{V}_o'$. Moreover (4.7) shows that for each $\mu \in \mathfrak{V}_o'$, the function $S_j(z,\mu)$ in z has the divisor $D_\mu - L = C_\mu$. Consequently we obtain (4.6). We note that the parameter variety $\mathfrak{V}_o'$ of the subset $\{C_\mu \mid \mu \in \mathfrak{V}_o'\}$ of $\mathfrak{M}'$ is *unique* in the following sense: Let $\mathfrak{V}^*$ be an open complex manifold and let $\mu^* \to C_{\mu^*}$ be a one-to-one analytic map of $\mathfrak{V}^*$ onto $\{C_\mu \mid \mu \in \mathfrak{V}_o'\}$. Then $\mu^* \to \mu = \mu(C_{\mu^*})$ is a bi-regular map of $\mathfrak{V}^*$ onto $\mathfrak{V}_o'$. This is an immediate consequence of the fact that the one-to-one mapping $\mu^* \to \mu = \mu(C_{\mu^*})$ of $\mathfrak{V}^*$ onto $\mathfrak{V}_o'$ is analytic.[31]

Now the *characteristic linear system* of the complete continuous system $\mathfrak{M}'$ on C may be defined as follows: We denote by $(\mu_1, \mu_2, \cdots, \mu_m)$ a system of local coordinates on M' with the center o and, for any "tangent vector" $u = (u_1, u_2, \cdots, u_m)$ of M' at o, we set

$$\partial_u = \sum_{\nu=1}^{m} u_\nu (\partial/\partial\mu_\nu).$$

Since the functions $S_j(z,o)$ vanish on $C = C_o$, we have

$$(\partial_u S_j(z,o))_C = b_{jk}(z,o)_C \cdot (\partial_u S_k(z,o))_C,$$

where $(\ )_C$ denotes restriction to C. On the other hand, the complex line bundle $B = [C] = [C_o]$ is defined by the system $\{b_{jk}(z,o)\}$. Hence

$$\phi_u : z \to \phi_u(z) = (z,\ \partial_u S_j(z,o)_C)$$

is a holomorphic section of the complex line bundle B_C over C, where $\partial_u S_j(z,o)_C$ stands for $(\partial_u S_j(z,o))_C$. Obviously we have

$$\phi_u = \sum u_\nu \phi_\nu, \quad \text{where } \phi_\nu(z) = (z, (\partial S_j(z,0)/\partial\mu_\nu)_C);$$

thus ϕ_u depends linearly on u. We associate with each tangent vector u such that $\phi_u \not\equiv 0$ the effective divisor $\bar{C}_u = (\phi_u)$ on C. Then the set $\{\bar{C}_u\}$ consisting of all such divisors $\bar{C}_u$ forms a linear system contained in $|B_C|$. This

[31] See footnotes 18 and 19.

linear system $\{\bar{C}_u\}$ is called the *characteristic linear system*[32] *of* $\mathfrak{M}'$ *on* C. The characteristic linear system $\{\bar{C}_u\}$ is uniquely determined by $\mathfrak{M}'$ and C. In fact, it is obvious that $\{\bar{C}_u\}$ is uniquely determined by the divisor $\mathfrak{D}'$ and is independent of the choice of the system of local coordinates $(\mu_1, \cdots, \mu_m)$ and of the system $\{S_j(z,\mu)\}$, while $\mathfrak{D}'$ is determined uniquely by the mapping $\mu \to C_\mu$ of $\mathfrak{V}_o'$ into $\mathfrak{M}'$. Hence the uniqueness of $\{\bar{C}_u\}$ follows from the uniqueness of the parameter variety $\mathfrak{V}_o'$ mentioned above.

THEOREM 4.1. *Let C be an effective divisor on V satisfying*

$$H^1(V, \Omega(C)) = 0. \tag{a}$$

Then C belongs to only one complete continuous system $\mathfrak{M}' = \{C_\mu \mid \mu \in M'\}$ and $\mu(C)$ is a simple point of the canonical parameter variety M' of $\mathfrak{M}'$. The dimension of the complete continuous system $\mathfrak{M}'$ is given by

$$\dim \mathfrak{M}' = \dim |C| + q, \tag{4.8}$$

where q is the dimension of the Picard variety $\mathfrak{P}$ attached to V. The system $\mathfrak{M}'$ contains all effective divisors $X \sim C$ on V satisfying $H^1(V, \Omega(X)) = 0$. If, moreover, C is a non-singular prime divisor, then the characteristic linear system of $\mathfrak{M}'$ on C is complete.

Proof. Let $\mathfrak{k}$, Λ, p, $\mathfrak{M}$, M have the same meaning as above and let $C = C_o$, $o = \mu(C)$. By Lemma 4.1, there exists a neighborhood $\mathfrak{U}$ of $p(o)$ on $\mathfrak{P}$ such that $M \cap p^{-1}(\mathfrak{U})$ is an analytic sub-bundle of $p^{-1}(\mathfrak{U})$ whose fibre is a linear variety of dimension $h = \dim |C|$. It follows that there exists only one irreducible component, say M' of M which meets with $p^{-1}(\mathfrak{U})$ and that

$$M' \cap p^{-1}(\mathfrak{U}) = M \cap p^{-1}(\mathfrak{U}). \tag{4.9}$$

Hence $C = C_o$ is contained in $\mathfrak{M}'$ but not in $\mathfrak{M}^{(l)}$, $l \geqq 2$, and $o = \mu(C)$ is a simple point of M'. Moreover $M' \cap p^{-1}(\mathfrak{U})$ is an open manifold of dimension $h + q$ and therefore we obtain (4.8). Clearly the projection $p(M^{(l)})$ of each component $M^{(l)}$ is an algebraic subvariety of $\mathfrak{P}$. Hence it follows from (4.9) that $p(M') = \mathfrak{P}$ and that $p(M^{(l)})$, $l \geqq 2$, are proper subvarieties of $\mathfrak{P}$. Now, given an effective divisor $X \sim C$ satisfying $H^1(V, \Omega(X)) = 0$, we write $X = C_\mu$, $\mu \in M$. The above argument shows that the component $M^{(l)} \ni \mu$ has the projection $p(M^{(l)}) = \mathfrak{P}$. Hence $M^{(l)}$ coincides with M' and therefore $X \in \mathfrak{M}'$.

---

[32] It is easy to see that our definition of characteristic linear systems coincides with the classical one. See Zariski [19], p. 78

Now, assuming that C is a non-singular prime divisor satisfying (a), we prove that the dimension of the characterstic linear system $\{\bar{C}_u\}$ of $\mathfrak{M}'$ on C is given by

$$\dim\{\bar{C}_u\} = h + q - 1, \qquad h = \dim |C|. \tag{4.10}$$

Obviously we may assume that

$$o = \mu(C) = (0, 1, 0, \cdots, 0)$$

in $p^{-1}(\mathfrak{U}) = \mathfrak{U} \times \mathfrak{S}_d$, where $\mathfrak{U} = \mathfrak{U}(0)$, $0 = p(o)$. It follows from (4.5) that a point $\mu \in \mathfrak{V}_o' \subset M' \cap \mathfrak{U}_o$ is written in the form

$$\mu = (P; 1, \eta_1, \cdots, \eta_h, \eta_{h+1}, \cdots, \eta_d),$$

where

$$\eta_\tau = -A_{0\tau}(P) - \sum_{\nu=1}^{h} A_{\nu\tau}(P)\eta_\nu, \qquad (\tau = h+1, \cdots, d).$$

Letting $P = \rho(t_1\tilde{\alpha}_1 + \cdots + t_q\tilde{\alpha}_q)$, for $P \in \mathfrak{U}$, we use

$$(\mu_1, \cdots, \mu_q, \mu_{q+1}, \cdots, \mu_{q+h}) = (t_1, \cdots, t_q, \eta_1, \cdots, \eta_h)$$

as the system of local coordinates on the neighborhood $\mathfrak{V}_o'$ of o on M'. The fibre coordinate $\psi_{\nu j}(z, P)$ of the section $\psi_\nu(P) = r_P{}^*\psi_\nu$ is given by $\psi_{\nu j}(z, P) = \psi_{\nu j 0}(z, P)$, as (3.19) shows. Letting $f_{jQkQ'}(z, P)$ have the same meaning as in Section 3, we have therefore

$$\psi_{\nu j}(z, P) = f_{j0k0}(z, P)\psi_{\nu k}(z, P). \tag{4.11}$$

We infer from (4.4) that, for each $\nu = 0, 1, \cdots, h$,

$$\chi_{\nu j}(z, P) = w_j(z)^{-1}\{\psi_{\nu j}(z, P) - \sum_{\tau=h+1}^{d} A_{\nu\tau}(P)\psi_{\tau j}(z, P)\}$$

is a holomorphic function on $U_j \times \mathfrak{U}$. Moreover it follows from (4.11) that

$$\chi_{\nu j}(z, P) = b_{jk}(z, P) \cdot \chi_{\nu k}(z, P), \tag{4.12}$$

where

$$b_{jk}(z, P) = f_{j0k0}(z, P) \cdot w_k(z)/w_j(z). \tag{4.13}$$

Thus, for each $P \in \mathfrak{U}$,

$$\chi_\nu(P) : z \to (z, \chi_{\nu j}(z)), \qquad \nu = 0, 1, \cdots, h$$

are holomorphic sections of $B + P$; obviously these sections $\chi_0, \chi_1, \cdots, \chi_h$ are linearly independent and therefore form a base of the linear space $\Gamma(B + P)$ of dimension $h + 1$. Now, using the explicit form (3.21) of the functions $R_j(z, \lambda)$, we obtain

$$(4.14)\qquad S_j(z,\mu) = R_j(z,\mu)/w_j(z) = \chi_{0j}(z,P) + \sum_{\nu=1}^{h} \mu_{q+\nu}\chi_{\nu j}(z,P).$$

Clearly we have $S_j(z,\mu) = b_{jk}(z,P)\cdot S_k(z,\mu)$. For an arbitrary tangent vector $u = (u_1, \cdots, u_q, u_{q+1}, \cdots, u_{q+h})$ of M' at o, we get, by (4.14),

$$(4.15)\qquad \partial_u S_j(z,o) = \partial_u \chi_{0j}(z,0) + \sum_{\nu=1}^{h} u_{q+\nu}\chi_{\nu j}(z,0)$$

where

$$\partial_u \chi_{0j}(z,0) = \sum_{k=1}^{q} u_k [\partial \chi_{0j}(z,P)/\partial t_k]_{t=0}.$$

In order to prove (4.10) it is sufficient to show that the holomorphic section

$$\phi_u\colon\ z \to \phi_u(z) = (z,\ \partial_u S_j(z,o)_C)$$

of B_C vanishes only if $u = 0$. Assume that $\partial_u S_j(z,o)_C = 0$. Then, since C is the divisor of $S_j(z,o)$ on U_j, each ratio

$$s_j(z) = \partial_u S_j(z,o)/S_j(z,o)$$

is a holomorphic function on U_j. On the other hand, it follows from (4.12) that

$$(4.16)\qquad \mathfrak{f}_u(z) = \sum_{\nu=1}^{h} u_{q+\nu}\chi_{\nu j}(z,0)/\chi_{0j}(z,0)$$

is a well defined meromorphic function on V. Dividing (4.15) by $S_j(z,o) = \chi_{0j}(z,0)$, we get therefore

$$(4.17)\qquad s_j(z) - \partial_u \log \chi_{0j}(z,0) = \mathfrak{f}_u(z), \qquad \text{in } U_j.$$

Applying (2.3) to the function $f(\hat{z},\tilde{\alpha};g,\tilde{\delta})$ defining the bundle Ξ, we infer, in the same manner as in the proof of (2.5), that

$$f_{joko}(z,P) = e_{jk}(z)\kappa_j\kappa_k^{-1}\cdot \exp\int_{z(j)}^{z(k)} \tilde{\alpha},$$

where $\{e_{jk}(z)\}$ is the system of transition functions defining the bundle $E = B + [L]$ and where

$$\kappa_j = \exp\int_{\hat{o}}^{\hat{z}(j)} \varpi^*\tilde{\alpha}.$$

Inserting this into (4.13), we obtain

$$b_{jk}(z,P) = b_{jk}(z,0)\,\kappa_j\kappa_k^{-1}\exp\int_{z(j)}^{z(k)} \tilde{\alpha}.$$

Using (4.12) we get therefore

$$\partial_u \log \chi_{0j}(z,0) - \partial_u \log \chi_{0k}(z,0) = \iota_j - \iota_k + \int_{z(j)}^{z(k)} \tilde{\alpha}_u,$$

where $\tilde{\alpha}_u = \sum_{k=1}^{q} u_k \tilde{\alpha}_k$ and where $\iota_j = \int_{\hat{o}}^{\hat{z}(j)} \varpi^* \tilde{\alpha}_u$. Combining this with (4.17) we obtain

$$s_j(s) - \iota_j - \int_{z(j)}^{z} \tilde{\alpha}_u = s_k(z) - \iota_k - \int_{z(k)}^{z} \tilde{\alpha}_u.$$

Thus $\rho(z) = s_j(z) - \iota_j - \int_{z(j)}^{z} \tilde{\alpha}_u$ is a well defined function of class C^∞ on V and $d\rho(z) = ds_j(z) - \tilde{\alpha}_u$. This implies that $ds_j = ds_k$ is a well defined holomorphic 1-form on V. Consequently, we have $ds_j(z) = 0$ and $\tilde{\alpha}_u = 0$. The equality $\tilde{\alpha}_u = 0$ implies that $u_1 = u_2 = \cdots = u_q = 0$. Hence we get $\partial_u \log \chi_{0j}(z,0) = 0$ and therefore, by (4.17), $d\mathfrak{f}_u(z) = ds_j(z) = 0$. Thus $\mathfrak{f}_u(z) = c_0$ is a constant on V and consequently by (4.16),

$$c_0 \chi_{0j}(z,0) + \sum_{\nu=1}^{h} u_{q+\nu} \chi_{\nu j}(z,0) = 0.$$

This implies that $u_{q+1} = u_{q+2} = \cdots = u_{q+h} = 0$. Thus we see that ϕ_u vanishes only if $u = 0$. This completes the proof of (4.10).

The characteristic linear system $\{\bar{C}_u\}$ of $\mathfrak{M}'$ on C is a subsystem of the complete linear system $|B_C|$. In order to prove that $\{\bar{C}_u\}$ coincides with $|B_C|$, we observe the exact sequence

$$0 \to \Omega \to \Omega(B) \xrightarrow{r_C} \Omega(B_C) \to 0.$$

The corresponding exact cohomology sequence

$$0 \to \boldsymbol{C} \to \Gamma(B) \to \Gamma(B_C) \to H^1(V,\Omega) \to \cdots$$

shows that

$$\dim |B_C| \leqq \dim |B| - 1 + \dim H^1(V,\Omega),$$

while we have $H^1(V,\Omega) \cong \bar{\mathfrak{N}}$, and so $\dim H^1(V,\Omega) = \dim \mathfrak{P} = q$. Consequently we obtain the *inequality*

$$\dim |B_C| \leqq \dim |C| + q - 1. \tag{4.18}$$

Comparing this with (4.10), we infer immediately that $\{\bar{C}_u\} = |B_C|$. Thus the characteristic linear system $\{\bar{C}_u\}$ of $\mathfrak{M}'$ on C is complete.

We note that a divisor C on V satisfies the above condition (a) if

$$c([C] - K) > 0 \tag{c}$$

holds. This is an immediate consequence of Theorem 1.2. Moreover it follows from a remark in Section 1 that the condition (c) is equivalent to

(c′) $|m(C-K)|$ is ample for sufficiently large integer m.

Another remark is that the condition (a) is actually weaker than (c). This can be easily seen by observing the simplest case where V is an *algebraic curve* of genus p. In fact, a general effective divisor C of degree $d \geqq p$ on V satisfies (a), while (c) holds for C if and only if $d > 2p-2$.

THEOREM 4.2. *Let $\mathfrak{M}' = \{C_\mu \mid \mu \in M'\}$ be a complete continuous system canonically parametrized by M'. Assume that*

$$H^1(V, \Omega(C_\mu)) = 0, \qquad \text{for all } C_\mu \in \mathfrak{M}'.$$

Then the parameter variety M' is an analytic fibre bundle over the Picard variety $\mathfrak{P}$ of V whose fibre is the projective space $\mathfrak{S}_h$ of dimension $h = \dim |C_\mu|$ and whose structure group is the projective transformation group. The canonical projection of M' onto $\mathfrak{P}$ is given by

$$p: \mu \to p(\mu) = [C_\mu - C_o], \qquad o \in M'.$$

Moreover $\mathfrak{M}'$ consists of all effective divisors $X \sim C_o$, $C_o \in \mathfrak{M}'$, on V.

Proof. Let $\mathfrak{k}$, Λ, p, $\mathfrak{M}$, M have the same meaning as above. We infer from Lemma 4.1 that, for each point $P = p(\mu)$, $\mu \in M'$, there exists a neighborhood $\mathfrak{U}(P)$ of P on $\mathfrak{P}$ such that $M \cap p^{-1}(\mathfrak{U}(P))$ is an analytic subbundle of $p^{-1}(\mathfrak{U}(P))$ whose fibre is a linear variety of dimension $h = \dim |C_\mu|$. It follows that

$$M' \cap p^{-1}(\mathfrak{U}(P)) = M \cap p^{-1}(\mathfrak{U}(P))$$

and therefore $p(M') \supset \mathfrak{U}(P)$, $P \in p(M')$, while $p(M')$ is a compact subset of $\mathfrak{P}$. Hence $p(M')$ coincides with $\mathfrak{P}$ and consequently M' is an analytic fibre bundle over $\mathfrak{P}$ whose fibre is $\mathfrak{S}_h$ and whose structure group is the projective transformation group. Moreover M' coincides with M and therefore $\mathfrak{M}'$ coincides with $\mathfrak{M}$, q. e. d.

In the special case where V is an *algebraic curve* of genus p, the *index of speciality* $i|C_\mu| = \dim H^1(V, \Omega(C_\mu))$ vanishes if the degree of the divisor C_μ is greater than $2p-2$. Therefore our result reduces in this case to a theorem of Chow[33] to the effect that the continuous system of all effective divisors of degree $d > 2p-2$ on the curve V is a projective bundle over the Jacobian variety attached to V. Thus our Theorem 4.2 may be regarded as a generalization of the above theorem of Chow to higher dimensional varieties.

PRINCETON UNIVERSITY AND THE INSTITUTE FOR ADVANCED STUDY.

[33] Chow [5].

REFERENCES.

[1] W. L. Baily, "On the quotient of an analytic manifold by a group of analytic homeomorphisms," *Proceedings of the National Academy of Sciences*, vol. 40 (1954), pp. 804-808.

[2] ———, *On the quotient of a complex analytic manifold by a discontinuous group of complex analytic self-homeomorphisms.* Thesis, Princeton (1955).

[3] S. Bochner and W. T. Martin, *Several complex variables*, Princeton, 1948.

[4] W. L. Chow, "On Picard varieties," *American Journal of Mathematics*, vol. 74 (1952), pp. 895-909.

[5] ———, "The Jacobian variety of an algebraic curve, *ibid.*, vol. 76 (1954), pp. 453-476.

[6] ———, "On complex analytic varieties, *ibid.*, to appear.

[7] J. Igusa, "On Picard varieties attached to algebraic varieties," *ibid.*, vol. 74 (1952), pp. 1-22.

[7a] ———, "On some problems in abstract algebraic geometry," to appear in the *Proceedings of the National Academy of Sciences.*

[8] K. Kodaira, "Some results in the transcendental theory of algebraic varieties," *Annals of Mathematics*, vol. 59 (1954), pp. 86-134.

[9] ———, "On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux," *Proceedings of the National Academy of Sciences*, vol. 39 (1953), pp. 865-868.

[10] ———, "On differential-geometric method in the theory of analytic stacks," *ibid.*, vol. 39 (1953), pp. 1268-1273.

[11] ———, "On Kähler varieties of restricted type," *Annals of Mathematics*, vol. 60 (1954), pp. 28-48.

[12] ——— and D. C. Spencer, "Groups of complex line bundles over compact Kähler varieties," *Proceedings of the National Academy of Sciences*, vol. 39 (1953), pp. 868-872.

[13] ——— and D. C. Spencer, "Divisor class groups on algebraic varieties," *ibid.*, vol. 39 (1953), pp. 873-877.

[14] ——— and D. C. Spencer, "On the varation of almost-complex structure," Mimeographed note, Princeton (1954).

[15] S. Lefschetz, "On certain numerical invariants of algebraic varieties with application to abelian varieties," *Transactions of the American Mathematical Society*, vol. 22 (1921), pp. 327-482.

[16] ———, *L'Analysis situs et la géometrie algébrique*, Paris, 1924.

[17] F. Severi, "Sul teorema fondamentale dei sistemi continui di curve sopra una superficie algebrica," *Annali di Matematicae*, vol. 23 (1944), pp. 149-181.

[18] A. Weil, "On Picard varieties," *American Journal of Mathematics*, vol. 74 (1952), pp. 865-894.

[19] O. Zariski, *Algebraic Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 3, No. 5, 1935.

[20] G. Zappa, "Sull'esistenza, sopra le superficie algebriche, di sistemi continui completi infiniti, la cui curva generica e a serie caratteristica incompleta," *Pont. Acad. Sci. Acta*, vol. 9 (1945), pp. 91-93.

FIBRE SYSTEMS OF JACOBIAN VARIETIES.*[1]

(II. Local Monodromy Groups of Fibre Systems)

By JUN-ICHI IGUSA.

---

1. Introduction. In our previous paper [2], we studied a fibre system of Jacobian varieties which is associated with a "general linear pencil" on a non-singular algebraic surface. The main result, there, was the compatibility of Chow's construction of Jacobian varieties and specialization of curves. In other words, degenerate fibres were shown to be completions of the generalized Jacobian varieties of the singular members of the pencil. In this paper, we shall investigate the "behavior of fibres along degenerate fibres." Speaking in the classical case, this can be explained as follows: Let $\{J_u\}$ be a fibre system of Jacobian varieties which is associated with a general linear pencil $\{C_u\}$ on the surface. If we take out "bad points" from the parameter sphere, we get a pointed sphere with non-trivial Poincaré group. This group operates on the fibre, and, in particular, we get a representation of the Poincaré group on the one-dimensional integer homology group of the fibre. This integer matric group we call the *monodromy group of the fibre system*. The monodromy group is a subgroup of the homogeneous modular group, and has many properties. In particular, it completely determines the integer homology groups of the surface [3]. In addition, if we take a sufficiently small neighborhood of one of the bad points, we can consider, so-to-speak, a *local monodromy group of the fibre system*. Now, if l is any prime number, the l-adic completions of these monodromy groups can be defined by a purely algebraic procedure. Furthermore, the same procedure, applied to the abstract case, gives rise what we call *abstract monodromy groups*. The main purpose of this paper is to determine the structure of the abstract local monodromy group of the fibre system for l different from the characteristic of the universal domain. An attentive reader will find an algebraic reproduction of the local theory of "normal functions," which were introduced by Poincaré, and fully investigated by Lefschetz (cf. [7], Chapters 6-7).

---

* Received February 27, 1956.

[1] This work was partially supported by a research grant of the National Science Foundation.

In fact, this paper is a preliminary step for the corresponding global theory, which we will present in our later papers.

2. Invariant and vanishing points. Let V be a non-singular algebraic surface in a projective space which is defined over an algebraically closed field k. Let $\{C\}$ be a linear pencil on V which is also defined over k. Let u be a generic point over k of the parameter straight line, say D, and let C be the corresponding member of the pencil. Let a be a point of D. We note, once for all, that any rational cycle over $k(u)$ in a projective space has a unique specialization over the specialization $u \to a$ with reference to k. Let C' be the specialization of C over the specialization $u \to a$ with reference to k. We assume that both C and C' are irreducible, and also that C is non-singular and C' has one and only one ordinary double point, say Q. In addition, we assume that Q is not a base point of the pencil, and that the pencil does have a base point, say A. We note that the points a, Q and A are all rational over k.

Now, let g be the genus of C. Then, g is equal to the arithmetic genus of C' [8]. Moreover, the true genus of C', i.e., the genus of the normalization C^* of C' over k is equal to $g-1$. We fix an integer d greater than $2g-2$ as a reference integer, and we construct the Jacobian variety J of C by Chow's method in a projective space, say P^N [1]. Then, the specialization J' of J over the specialization $C \to C'$ with reference to k is an irreducible variety, and contains the generalized Jacobian variety $(J')_0$ of C' which is constructed by Chow's method for the same d [2]. We can introduce group laws in J and in $(J')_0$ by taking $d \cdot A$ as reference divisor, and then, the laws are compatible with specialization. If J^* is the Jacobian variety of C^*, then, $(J')_0$ is an extension by J^* of the multiplicative group G_m of the universal domain [4]. Moreover, $J' - (J')_0$ is exactly the singular locus of J', and it is biregularly equivalent with J^*. We shall consider another group laws in J and in $(J')_0$ which are also compatible with specialization; we start by proving the following theorem:

THEOREM 1. *Let t be a rational point of J over $k(u)$. Then, the specialization, say t', of t over the specialization $u \to a$ with reference to k is a simple point of J'.*

Proof. If $\mathfrak{a}$ is a divisor of degree d on C, we say that a point s of J corresponds to $\mathfrak{a}$, and vice versa, if s is the Chow point of the variety of the complete linear system $|\mathfrak{a}|$ on C; similarly for C'. We note that this correspondence has nothing to do with group laws in J and in $(J')_0$. In

addition, the correspondence is compatible with specialization [2]. Since C contains a rational point over k, we can find a positive divisor $\mathfrak{m}$ of degree d which is rational over $k(u)$, and which corresponds to t [1]. Let $\mathfrak{m}_0$ be the fixed part of $\mathfrak{m}$, i.e., the partial sum of $\mathfrak{m}$ of base points of the linear pencil; put $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$. Then, $\mathfrak{m}_1$ is a rational cycle of V over $k(u)$, and hence there exists a rational cycle $\bar{X}$ over k of the product $D \times V$ such that $\bar{X} \cdot (u \times . V) = u \times \mathfrak{m}_1$. Here, the intersection-product is taken on $D \times V$ [5, p. 204]. Since $\bar{X}$ and $\mathrm{pr}_2(\bar{X})$ correspond biregularly except the base points of the pencil, by a simple calculus of cycles we get $\mathrm{pr}_2(\bar{X}) \cdot C = \mathfrak{m}_1$. Here, the intersection-product is taken on V. Since Q is simple on V, we can replace $\mathrm{pr}_2(\bar{X})$ by a rational divisor Y of V over k which is linearly equivalent to $\mathrm{pr}_2(\bar{X})$, and which does not pass through Q. Then, $\mathfrak{m}_0 + Y \cdot C$ is a rational divisor of C over $k(u)$ which is linearly equivalent to $\mathfrak{m}$ on C. Moreover, $\mathfrak{m}_0 + Y \cdot C'$ is the unique specialization of $\mathfrak{m}_0 + Y \cdot C$ over the specialization $u \to a$ with reference to k. Since $\mathfrak{m}_0 + Y \cdot C$ corresponds to t, we see that $\mathfrak{m}_0 + Y \cdot C'$ corresponds to t'. Here, $\mathfrak{m}_0 + Y \cdot C'$ is free from Q. q.e.d.

As the above proof shows, for any rational point t of J over $k(u)$, we can find a rational divisor $\mathfrak{a}$ of C of degree d over $k(u)$ which corresponds to t, and such that if $\mathfrak{a}_0 - \mathfrak{a}_1$ is the reduced expression of $\mathfrak{a}$ as a difference of positive divisors, the specializations of $\mathfrak{a}_0$ and $\mathfrak{a}_1$ over the specialization $u \to a$ with reference to k are both free from Q. In this connection, we state here the following assertion:

SUPPLEMENT. *Let $\mathfrak{t}$ be a rational and positive divisor of C over $k(u)$ such that the specialization, say $\mathfrak{t}'$, of $\mathfrak{t}$ over the specialization $u \to a$ with reference to k is free from Q. Then, if m is a sufficiently large integer, $\mathfrak{t}$ is a partial intersection by a hypersurface F of order m in the ambient space of C which is rational over k, and which does not pass through Q.*

Let $\mathfrak{t} = \mathfrak{t}_0 + \mathfrak{t}_1$ be the decomposition of $\mathfrak{t}$ into the fixed and variable parts; let $\mathfrak{t}_1'$ be the specialization of $\mathfrak{t}_1$ over the specialization $u \to a$ with reference to k. We represent $\mathfrak{t}_1$ as an intersection-product of C and a positive divisor X of V which is rational over k. Then, by the uniqueness of the specialization we get $X \cdot C' = \mathfrak{t}_1'$. Since $\mathfrak{t}' = \mathfrak{t}_0 + \mathfrak{t}_1'$ is free from Q, we see that X does not pass through Q. Now, by the method of projecting cones we can represent X as a partial intersection of V by a hypersurface F_0 which is rational over k. In addition, we can take F_0 so that it does not pass through Q. Let F_1 be a hypersurface which is rational over k, and which passes through $\mathfrak{t}_0$ but not through Q. If m is at least equal to the order

of F_0+F_1, we have only to add a suitable hypersurface to F_0+F_1 so that we get a hypersurface F of order m.

Now, coming back to our original problem, we shall prove the following assertion:

LEMMA 1. *Let G be a commutative group variety which is defined, together with its group law, over a field K; let t be a point of G. Then, $x\cdot y=x+y+t$ is a group law in G over K if and only if t is rational over K. Moreover, if G is an Abelian variety, every group law in G over K can be so obtained.*

Proof. The first assertion is trivial. Assume that G is an Abelian variety over K. Let ϕ be a new group law in G over K. Then, we can find an automorphism λ of G over K with respect to the old group law in G, and a rational point t of G over K such that $\phi(x,y)=\lambda(x+y)+t$ for x and y on G [6, p. 34]. The associativity of the law implies $\lambda^2=\lambda$, whence $\lambda=\delta_G$, the identity automorphism of G. q.e.d.

We pick an arbitrary rational point t of J over $k(u)$; let t' be the specialization of t over the specialization $u\to a$ with reference to k. Then, by Theorem 1 and by Lemma 1 we can introduce group laws in J and in $(J')_0$ over $k(u)$ and over k, respectively, such that $-t$ and $-t'$ become new neutral elements. Henceforth, we shall denote $-t$ by 0 and $-t'$ by 0'. We note that the group laws are compatible with specialization [2].

Let n be a positive integer which is not divisible by the characteristic p of the universal domain. Then, δ being the identity automorphism of J, we denote by T the graph of $n\cdot\delta$ in the product $J\times J$. As we know, if x is an arbitrary point of J, the cycle inverse $(n\cdot\delta)^{-1}(x)=\mathrm{pr}_1[T\cdot(J\times x)]$ is defined, and consists of n^{2g} distinct points. In fact, this is a general theorem which holds for any Abelian variety [5, p. 127]. Similarly, let δ' be the identity automorphism of $(J')_0$, and let $(T')_0$ be the graph of $n\cdot\delta'$ in $(J')_0\times(J')_0$. With these notations, we shall prove the following lemma:

LEMMA 2. *As an abstract group $(J')_0$ splits over G_m, i.e., $(J')_0=G_m\times((J')_0/G_m)$. Moreover, the cycle inverse*

$$(n\cdot\delta')^{-1}(x)=\mathrm{pr}_1[(T')_0\cdot((J')_0\times x)]$$

is defined for every point x of $(J')_0$, and consists of n^{2g-1} distinct points.

Proof. The first assertion is a consequence of a general theorem on Abelian groups, because G_m is "infinitely divisible."[2] Then, the facts that

[2] A precise statement of the theorem is the following: *Let G be an Abelian group, and let H be it subgroup which is infinitely divisible. Then, the extension G of H splits over H.*

the cycle inverse $(n\cdot\delta')^{-1}(x)$ is defined for every point x of $(J')_0$ and that the number of distinct points in it is equal to n^{2g-1} follow from the corresponding properties of the factors of $(J')_0$. It is, therefore, sufficient to show that the homomorphism $n\cdot\delta'$ is separable. A usual proof of this for any commutative group variety G, and for its n-times the identity automorphism δ_G is the following: We know that the ring of endomorphisms of G has a representation on th space of forms of Maurer-Cartan. The endomorphism $n\cdot\delta_G$ is represented by the n-times the unit matrix, and none of the forms of Maurer-Cartan becomes zero by multiplication of n. Since these forms generate the space of linear differential forms over the function-field, if x is a generic point of G over one of its fields of definition, say K, there is no derivation of $K(x)$ over $K(n\cdot x)$ except, of course, the trivial one. Hence $K(x)$ is separably algebraic over $K(n\cdot x)$. q. e. d.

Now, let T' be the closure of $(T')_0$ with respect to the Zariski topology. Then, from Lemma 2 we get $\mathrm{pr}_2(T')=n^{2g-1}\cdot J'$. On the other hand, by our previous remark we have $\mathrm{pr}_2(T)=n^{2g}\cdot J$. Therefore, if we specialize T over the specialization $u\to a$ with reference to k, we get T' and something more. The next lemma concerns about this:

LEMMA 3. *The specialization, say $\bar{T}$, of T over the specialization $u\to a$ with reference to k is of the form $\bar{T}=T'+T''$. Here, T'' is a positive cycle of the product $P^N\times P^N$ each component of which has a singular subvariety of J' as its projection on the first factor.*

Proof. Let $x'\times y'$ be a generic point over k of some component, say W, of $\bar{T}$. Then, we can find a generic point $x\times n\cdot x$ of T over $k(u)$ such that $x'\times y'$ is a specialization of $x\times n\cdot x$ over the specialization $u\to a$ with reference to k. If x' is a simple point of J', since the group laws are compatible with specialization, we have $y'=n\cdot x'$. In other words, if the projection of W on the first factor is not contained in the singular locus of J', then, W coincides with T'. Since $\mathrm{pr}_1(T')=J'$ is the specialization of $\mathrm{pr}_1(T)=J$ over the specialization $u\to a$ with reference to k, we see that $\bar{T}$ does contain T', and in fact only once. q. e. d.

It is a simple matter to determine all components of T''. We get just what is expected. However, since we shall not make use of such information, we do not discuss it here.

Now, let r be a rational point of J over $k(u)$, and let r' be the specialization of r over the specialization $u\to a$ with reference to k. Then, by Theorem 1 the point r' is simple on J'. We shall denote the 0-cycles $(n\cdot\delta)^{-1}(r)$ and $(n\cdot\delta')^{-1}(r')$ by w and w'. We remark that w is rational over $k(u)$, hence it has a unique specialization over the specialization $u\to a$ with reference to k. With this in mind, we shall prove the following theorem:

THEOREM 2. *The specialization, say $\bar{w}$, of w over the specialization $u \to a$ with reference to k is of the form $\bar{w} = w' + w''$. Here, w'' is a positive cycle of P^N each component of which is a multiple point of J'.*

Proof. We first note that w and w' are defined, respectively, as $\mathrm{pr}_1[T \cdot (J \times r)]$ and $\mathrm{pr}_1[(T')_0 \cdot ((J')_0 \times r')]$. Let s' be an arbitrary simple point of J'. Then, $s' \times r'$ is not contained in any component of T'' of Lemma 3. Since $(T' + T'', J', r')$ is the specialization of (T, J, r) over the specialization $u \to a$ with reference to k, by the "principle of conservation of number"[3] $s' \times r'$ appears in $\bar{w} \times r'$ and in $T' \cdot (J' \times r')$ exactly the same number of times. However, we have $T' \cdot (J' \times r') = (T')_0 \cdot ((J')_0 \times r')$, the intersection-products being taken on $J' \times J'$ and on $(J')_0 \times (J')_0$. This is nothing but $w' \times r'$. q.e.d.

Let $k[[u-a]]$ be the completion of the specialization-ring of a in $k(u)$, and let $k((u-a))$ be its quotient-field. We note that $k[[u-a]]$ is a ring of formal power-series in one letter over k. In the following, we consider $k(u)$ as a subfield of the "abstract field" $k((u-a))$. Let I be an isomorphism over $k(u)$ of the algebraic closure of $k(u)$ into some, but fixed, algebraic closure of $k((u-a))$. Let (w) be an ordered set of points in w, and let $I(w)$ be the transform of (w) by I. Then, the specialization of $I(w)$ at the center of $k[[u-a]]$ is an ordered set of points in $\bar{w}$, because it is a specialization of (w) over the specialiaztion $u \to a$ with reference to k. In particular, (w) contains n^{2g-1} points, say $(s_1, s_2, \cdots)$, such that the specialization of $I(s_1, s_2, \cdots)$ at the center of $k[[u-a]]$ is an ordered set of points of w'. Then, by generalized "Hensel's lemma" each $I(s_i)$ is rational over $k((u-a))$ (cf. [5], Theorem 2, p. 57). We note that the set $(s_1, s_2, \cdots)$ is uniquely determined up to a permutation of elements by I. Now, assume that r and r' are the neutral elements 0 and 0′ of J and $(J')_0$. Then, the points $s_1, s_2, \cdots$ form an Abelian group, which is a direct product of $2g-1$ cyclic groups of order n. We call it the *group of invariant points of order n along I*. In fact, each $I(s_i)$ is "invariant" by any automorphism of the algebraic closure of $k((u-a))$ over $k((u-a))$. On the other hand, this group contains a particular set of n points, say $t_1, t_2, \cdots$, such that $I(t_i)$ is specialized to a point of G_m at the center of $k[[u-a]]$. We note that the

---

[3] This famous principle has various formulations. Here, it is used with the following content: *Let X and Y be two positive cycles on a variety V in a projective space such that $\dim(X) + \dim(Y) = \dim(V)$; let (V', X', Y') be a specialization of (V, X, Y) over a field k, and assume that V' is a variety. If the intersection-products $X \cdot Y$ and $X' \cdot Y'$ are defined on V and on V', a simple point P' of V' appears $i(X' \cdot Y', P'; V')$-times in any specialization of $X \cdot Y$ over the specialization $(V, X, Y) \to (V', X', Y')$ with reference to k.*

points $t_1, t_2, \cdots$ form a cyclic group of order n, and we call it the *group of vanishing points of order n along I*. In fact, G_m is the set of those points of $(J')_0$ which "vanish" by projection to the Jacobian variety J^*!

3. Non-invariant points. In the previous section, we introduced invariant and vanishing points. We shall now examine "non-invariant" points; the following is our main theorem:

THEOREM 3. *Let r be a rational point of J over $k(u)$; let x and y be two points of J such that $n \cdot x = n \cdot y = r$, and such that $I(x)$ and $I(y)$ are conjugate over $k((u-a))$. Then, $x-y$ is a vanishing point of order n along I.*

Proof. Since the theorem is trivial for $x = y$, we shall exclude this case from the beginning. Then, by Theorem 2 the specialization, say x', of $I(x)$ at the center of $k[[u-a]]$ is a multiple point of J'. Also, by Hensel's lemma (x', x') is the specialization of $(I(x), I(y))$ at the center of $k[[u-a]]$. Now, take an integer m such that the trace $\mathfrak{L}_m$ on C of the complete linear system of hypersurfaces of order m in the ambient space is complete, and of dimension equal to $r = \mathrm{ord}(C) \cdot m - g$. We also assume that the trace $\mathfrak{L}_m'$ on C' of the same linear system of hypersurfaces is of dimension r [8]. If m satisfies these conditions, we call it to be sufficiently large. We pick rational divisors $\mathfrak{a}$ and $\mathfrak{s}$ of C of degree d over $k(u)$ which correspond to 0 and r as remarked next to Theorem 1; put $\mathfrak{a} = \mathfrak{a}_0 - \mathfrak{a}_1$ and $\mathfrak{s} = \mathfrak{s}_0 - \mathfrak{s}_1$. We then take a hypersurface of a sufficiently large order, say m, for $\mathfrak{s}_0 + (n-1) \cdot \mathfrak{a}_0$ as stated in the Supplement of Theorem 1. If $\mathfrak{r}_1$ is the sum of $\mathfrak{s}_1 + (n-1) \cdot \mathfrak{a}_1$ and the residual intersection on C, then, $\mathfrak{r}_1$ is rational over $k(u)$, and the specialization, say $\mathfrak{r}_1'$, of $\mathfrak{r}_1$ over the specialization $u \to a$ with reference to k is free from Q. Consider the residual linear system $\mathfrak{L}_m - \mathfrak{r}_1$. The set of positive divisors of degree d on C such that their n-times belong to $\mathfrak{L}_m - \mathfrak{r}_1$ splits into n^{2g} distinct complete linear systems on C, and these linear systems correspond in a one-to-one way to points of J such as x and y. Let $|\mathfrak{m}|$ be the complete linear system which corresponds to x. We may assume that $\mathfrak{m}$ is algebraic over $k(u)$, and such that the specializations of $\mathfrak{m}$ over the specialization $(u, x) \to (a, x')$ with reference to k contain Q once and only once. In fact, let $\mathfrak{M}$ and $\mathfrak{M}'$ be the varieties which have x and x' as their Chow points. Then, we can find a subspace of co-dimension $d-g$, in the ambient space of $\mathfrak{M}$ and $\mathfrak{M}'$, which intersects properly with $\mathfrak{M}$ and $\mathfrak{M}'$. In addition, we can assume that the subspace is defined over k, and also that it does not intersect with the closed subset of $\mathfrak{M}'$ whose points represent those cycles which contain Q at least twice [2]. If $\mathfrak{m}$ is one of the divisors of C

which correspond to the intersection of $\mathfrak{M}$ and the subspace, $\mathfrak{m}$ satisfies our requirements. Now, there are many hypersurfaces of order m which intersect C at $n \cdot \mathfrak{m} + \mathfrak{r}_1$. However, since C and C' have the same arithmetic genus, we can pick one, say F_1, which is algebraic over $k(u)$, and such that F_1 has a unique specialization over any specialization $(u, \mathfrak{m}) \to (a, \mathfrak{m}')$ with reference to k, and such that this specialization does not contain C'. In fact, the complete linear system of hypersurfaces of order m in the ambient space of C and C' gives rise a regular rational map of this projective space into another projective space, say P. Let P^r and P'^r be the smallest subspaces of P which contain the images of C and C', respectively. Take a subspace P^t of P of co-dimension $r+1$ which spans P with P^r, and also with P'^r. Since P is defined over the prime field, we can assume P^t to be algebraic over the prime field. On the other hand, since $n \cdot \mathfrak{m} + \mathfrak{r}_1$ is rational over $k(u, \mathfrak{m})$, the hyperplane P^{r-1} of P^r, which corresponds to $n \cdot \mathfrak{m} + \mathfrak{r}_1$, is purely inseparable over $k(u, \mathfrak{m})$. If F_1 is the hypersurface of the original projective space which is represented by the join of P^{r-1} and P^t, then, F_1 satisfies our demands. Let F' be the specialization of $I(F_1)$ at the center of $k[[u-a]]$. We note that $I(F_1)$ has a meaning, because F_1 is algebraic over $k(u)$. Let $(I(y), \bar{\mathfrak{m}}, \bar{F}_1)$ be a conjugate of $(I(x), I(\mathfrak{m}), I(F_1))$ over $k((u-a))$. Then, we can find a positive divisor $\mathfrak{n}$ of degree d on C and a hypersurface F_2 of order m such that $\bar{\mathfrak{m}} = I(\mathfrak{n})$ and $\bar{F}_1 = I(F_2)$. Since $(\mathfrak{n}, F_2)$ is a conjugate of $(\mathfrak{m}, F_1)$ over $k(u)$, we see that F_2 intersects C at $n \cdot \mathfrak{n} + \mathfrak{r}_1$. Since $(y, \mathfrak{n})$ is a conjugate of $(x, \mathfrak{m})$ over $k(u)$, we see that $\mathfrak{n}$ corresponds to y. Also, by Hensel's lemma $(\mathfrak{m}', \mathfrak{m}')$, say, and (F', F') are the specializations of $(I(\mathfrak{m}), I(\mathfrak{n}))$ and $(I(F_1), I(F_2))$ at the center of $k[[u-a]]$. This will become a key point in our later argument. Now, take a hypersurface, say G_1, of a sufficiently large order m', which passes through $\mathfrak{m} + \mathfrak{a}_0$. We take G_1 to be algebraic over $k(u)$, and such that the specialization, say G_1', of $I(G_1)$ at the center of $k[[u-a]]$ does not contain C'. Also, we assume that G_1' contains Q only twice. In fact, these can be managed, more or less, by similar considerations as before. Let $\mathfrak{r}_2$ be the sum of $\mathfrak{a}_1$ and the residual intersection of G_1 on C. Then, the residual linear system $\mathfrak{L}_{m'} - (\mathfrak{n} + \mathfrak{r}_2)$ exists, and it is the complete linear system which corresponds to $x - y$. We shall show that the specialization, say $(x-y)'$, of $I(x-y)$ at the center of $k[[u-a]]$ is a simple point of J'. Otherwise, we can pick $\mathfrak{p}$ from $\mathfrak{L}_{m'} - (\mathfrak{n} + \mathfrak{r}_2)$ such that $\mathfrak{p}$ is algebraic over $k(u)$, and such that the specialization $\mathfrak{p}'$ of $I(\mathfrak{p})$ at the center of $k[[u-a]]$ contains Q once and only once. Let G_2 be a hypersurface of order m' which is algebraic over $k(u)$, and which intersects C at $\mathfrak{p} + \mathfrak{n} + \mathfrak{r}_2$. We may assume that the specialization, say G_2', of $I(G_2)$ at the center of $k[[u-a]]$ does not contain

C'. Finally, let H_1 be a hypersurface of a sufficiently large order, say m'', for $n \cdot \mathfrak{a}_0$ as stated in the Supplement of Theorem 1. If $\mathfrak{r}_3$ is the sum of $n \cdot \mathfrak{a}_1$ and the residual intersection on C, then, $\mathfrak{r}_3$ is rational over $k(u)$, and the specialization, say $\mathfrak{r}_3'$, of $\mathfrak{r}_3$ over the specialization $u \to a$ with reference to k is free from Q. Let H_2 be a hypersurface of order m'' which is algebraic over $k(u)$, and which intersects C at $n \cdot \mathfrak{p} + \mathfrak{r}_3$. We may assume that the specialization, say H_2', of $I(H_2)$ at the center of $k[[u-a]]$ does not contain C'. Then, by our construction two hypersurfaces $F_1 + n \cdot G_2 + H_1$ and $F_2 + n \cdot G_1 + H_2$ have the same intersection with C. Therefore, the specializations $F' + n \cdot G_2' + H_1$ and $F' + n \cdot G_1' + H_2'$ have the same intersection with C'. More precisely, if C^* is the normalization of C' over k, and if f is the corresponding regular rational map of C^* into the ambient space of C', we have $f^{-1}(F' + n \cdot G_2' + H_1) = f^{-1}(F' + n \cdot G_1' + H_2')$, and hence $n \cdot f^{-1}(G_2') + f^{-1}(H_1) = n \cdot f^{-1}(G_1') + f^{-1}(H_2')$. Let Q_1^* and Q_2^* be the points of C^* which correspond to Q. Let c_1 and c_2 be the coefficients of Q_1^* and Q_2^* in the reduced expression of the divisor of C^* just obtained. Since H_1 does not pass through Q, if we use the first expression, we conclude that both c_1 and c_2 are multiples of n. On the other hand, H_2' intersects C' at $n \cdot \mathfrak{p}' + \mathfrak{r}_3'$, and $\mathfrak{p}'$ contains Q once and only once, while $\mathfrak{r}_3'$ is free from Q. Since $f^{-1}(H_2')$ must contain both Q_1^* and Q_2^*, we conclude from the second expression that neither c_1 nor c_2 is divisible by n. Thus, we get a contradiction, and this contradiction is derived from the assumption that $(x-y)'$ is a multiple point of J'. In other words, we have shown that $(x-y)'$ is a simple point of J'. Then, our previous argument goes through under the modification that $\mathfrak{p}'$ is free from Q. Since $\mathfrak{m}'$ contains Q only once and G_1' does only twice, we conclude that $f^{-1}(G_2') - f^{-1}(G_1') = \mathfrak{p}^* - \mathfrak{a}^*$. Here, $\mathfrak{p}^*$ and $\mathfrak{a}^*$ correspond to $\mathfrak{p}'$ and the specialization, say $\mathfrak{a}'$ of $\mathfrak{a}$ over the specialization $u \to a$ with reference to k. Since the left side is linearly equivalent to zero on C^*, so is also the right side, i.e., $(x-y)'$ belongs to G_m. Therefore, by definition $x-y$ is a vanishing point of order n along I. q.e.d.

Remark. If the linear equivalence class of $Q_1^* - Q_2^*$ on C^* is of order at least equal to n, we can prove the following assertion: *Let x and y be two points of J such that $n \cdot x = n \cdot y = r$, and such that (x', x'), say, is a specialization of (x, y) over the specialization $u \to a$ with reference to k. Then, any specialization of $x - y$ over this specialization is a point of G_m.* A proof of this assertion can be extracted from the proof we gave for Theorem 3.

4. A bilinear relation. In this section, we shall show that invariant points and vanishing points are connected by a bilinear relation. Let U be,

for a moment, an arbitrary complete curve. Then, for any numerical function f on U, we can define a divisor (f), and a divisor in the sense of valuation theory. The latter is defined as the regular image on U of the divisor of the function induced by f on the non-singular model of U. On the other hand, if $\sum_{\mathfrak{a}} a_{\mathfrak{a}} \cdot P_{\mathfrak{a}}$ is a reduced expression of a divisor $\mathfrak{a}$ on U, and if (f) does not contain any $P_{\mathfrak{a}}$, we define $f(\mathfrak{a})$ by $\prod_{\mathfrak{a}} f(P_{\mathfrak{a}})^{a_{\mathfrak{a}}}$. If, moreover, $f(\mathfrak{b})$ is defined for a divisor $\mathfrak{b}$ of U, then, $f(\mathfrak{a}+\mathfrak{b})$ is defined, and we have $f(\mathfrak{a}+\mathfrak{b}) = f(\mathfrak{a}) \cdot f(\mathfrak{b})$. These being remarked, we shall prove the following lemma:

LEMMA 4. *Let f and h be two functions on U such that their divisors in the sense of valuation theory coincide, respectively, with (f) and (h). If, moreover, either $f((h))$ or $h((f))$ is defined, so is the other, and they are equal.*

Proof. We first note that nothing is changed by passing to the non-singular model of U. Therefore, we can assume from the beginning that U is non-singular. Also, the first assertion of the lemma is trivial. Let K be a field over which U, f and h are defined; let t be a variable over K. We shall denote by Γ_f the graph of f in the product $U \times D$; similarly for Γ_h. Here, D is, of course, a projective straight line. We define $(h)_t$ by $\Gamma_h \cdot (U \times (t)) = (h)_t \times (t)$, and then $(h)_t$ is prime rational over $K(t)$. Hence, $f((h)_t)$ is rational over $K(t)$, and we can find a function θ on D, which is defined over K, such that $f((h)_t) = \theta(t)$. Then, the divisor of θ is given by $(\theta) = \mathrm{pr}_2[\Gamma_h \cdot ((f) \times D)]$ [5, p. 231]. More explicitly, if we write (f) in the form $\sum_{\mathfrak{a}} a_{\mathfrak{a}} \cdot P_{\mathfrak{a}}$, we get $(\theta) = \sum_{\mathfrak{a}} a_{\mathfrak{a}} \cdot h(P_{\mathfrak{a}})$. Therefore, $\theta(t)$ is of the form $c \cdot \prod_{\mathfrak{a}} (t - h(P_{\mathfrak{a}}))^{a_{\mathfrak{a}}}$ with a certain element c in K, not equal to 0. If we specialize t to 0 and ∞, we get $f((h)_0) = c \cdot h((f))$ and $f((h)_\infty) = c$, hence $f((h)) = h((f))$. q. e. d.

Now, let $\mathfrak{a}$ and $\mathfrak{b}$ be two divisors on the non-singular curve C such that $n \cdot \mathfrak{a}$ and $n \cdot \mathfrak{b}$ are, respectively, divisors of functions f and h on C. Then, if either $f(\mathfrak{b})$ or $h(\mathfrak{a})$ is defined, so is the other, and the expression

$$(\mathfrak{a}, \mathfrak{b}; n) = h(\mathfrak{a}) \cdot f(\mathfrak{b})^{-1}$$

is an n-th root of unity by Lemma 4. Furthermore, the same lemma implies that $(\mathfrak{a}, \mathfrak{b}; n)$ depends only on the linear equivalence classes of $\mathfrak{a}$ and $\mathfrak{b}$ on C. Since any linear equivalence class contains a divisor whose components are all generic over a given field, $(\mathfrak{a}, \mathfrak{b}; n)$ induces a multiplication of the group of divisor classes of order n on C with itself.

On the other hand, let ϕ be a canonical function of C which is defined

over $k(u)$. Let t be an arbitrary point of the Jacobian variety J of C, and let $M_1, \cdots, M_{g-1}$ be independent generic points of C over $k(u,t)$. Then, Θ_t is defined as the locus over $k(u,t)$ of the point $\sum_{i=1}^{g-1}\phi(M_i)+t$, and Θ_0 is denoted simply by Θ. Also, if $\mathfrak{b}=\sum_\beta b_\beta\cdot Q_\beta$ is an arbitrary divisor of C, the point $\sum_\beta b_\beta\cdot\phi(Q_\beta)$ of J is denoted by $S[\phi(\mathfrak{b})]$. Put $t=S[\phi(\mathfrak{b})]$. Then, we have $n\cdot t=0$ if and only if $n\cdot\mathfrak{b}$ is linearly equivalent to zero on C. In this case, if X is a divisor of J which is linearly equivalent to $\Theta_t-\Theta$, we can find a function ϕ on J such that $(\phi)=n\cdot X$. Moreover, if x is a generic point of J over a field of definition K of ϕ, there exists a function ψ on J, which is defined over K, such that $\phi(n\cdot x)=\psi(x)^n$ [6, p. 150]. Now, let s be another point of J of order n. Then, $\psi(x+s)\cdot\psi(x)^{-1}$ is an n-th root of unity, which is denoted by $e_n(s,t)$:

$$\psi(x+s)=e_n(s,t)\cdot\psi(x).$$

As we can see, $e_n(s,t)$ is independent of the choice of X in the linear equivalence class of $\Theta_t-\Theta$. The two n-th roots of unity so obtained are related as follows:

PROPOSITION (Weil). *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two divisors of C, and put $s=S[\phi(\mathfrak{a})]$ and $t=S[\phi(\mathfrak{b})]$. If both s and t are of order n, and if $(\mathfrak{a},\mathfrak{b};n)$ is defined, we have $(\mathfrak{a},\mathfrak{b};n)=e_n(s,t)$.*

Proof.[4] Since $n\cdot\mathfrak{b}$ is linearly equivalent to zero on C, we can find a function h on C such that $(h)=n\cdot\mathfrak{b}$. Pick a divisor $\mathfrak{m}_0$ of degree g on C which has no point in common with $\mathfrak{b}$. We take an extension K of $k(u)$ over which $\mathfrak{b}$, $\mathfrak{m}_0$ and s are all rational. Let $M_1,\cdots,M_g$ be independent generic points of C over K, and put $\mathfrak{m}_1=\sum_{i=1}^{g}M_i$ and $x=S[\phi(\mathfrak{m}_1-\mathfrak{m}_0)]$. Then, $h(\mathfrak{m}_1-\mathfrak{m}_0)$ is defined, and it is rational over $K(\mathfrak{m}_1)=K(x)$. Therefore, we can find a function ϕ on J, which is defined over K, such that $\phi(x)=h(\mathfrak{m}_1-\mathfrak{m}_0)$. If $\sum_\beta b_\beta\cdot Q_\beta$ is the reduced expression of $\mathfrak{b}$, and if we put $X=\sum_\beta b_\beta\cdot\Theta_{\phi(Q_\beta)}$, we have $(\phi)=n\cdot X$ (cf. [6], pp. 115-116). Moreover, X is linearly equivalent to $\Theta_t-\Theta$ [6, pp. 105-106]. On the other hand, by Riemann-Roch theorem the linear equivalence class of $\mathfrak{m}_0+n\cdot(\mathfrak{m}_1-\mathfrak{m}_0)$ contains at least one positive divisor, say $\mathfrak{m}_n$. Since $S[\phi(\mathfrak{m}_n-\mathfrak{m}_0)]=n\cdot x$ is a generic point of J over K, we conclude that the components of $\mathfrak{m}_n$ are

[4] This proof was communicated to us by Professor André Weil on February 6, 1956. The proposition is interesting, and it simplifies the proof of Theorem 4. In fact, our original proof was direct, and it was much longer than the present one.

g independent generic points of C over K. In particular, $\mathfrak{m}_n$ is unique, and hence it is rational over $K(x)$. Therefore, we can find a function F on C, which is defined over $K(x)$, such that $(F) = \mathfrak{m}_n - \mathfrak{m}_0 - n \cdot (\mathfrak{m}_1 - \mathfrak{m}_0)$. Let $(\mathfrak{m}_1', F')$ be the transform of $(\mathfrak{m}_1, F)$ by the automorphism of $K(x)$ over $K(n \cdot x)$ mapping x to $x + s$. Then, we have $S[\phi(\mathfrak{m}_1' - \mathfrak{m}_0)] = x + s$, i.e., $S[\phi(\mathfrak{m}_1' - \mathfrak{m}_1)] = s$. In other words, $\mathfrak{m}_1' - \mathfrak{m}_1$ is linearly equivalent to $\mathfrak{a}$. Moreover, we have $(F') = \mathfrak{m}_n - \mathfrak{m}_0 - n \cdot (\mathfrak{m}_1' - \mathfrak{m}_0)$. Therefore, if we define a function f on C by $f = F \cdot [F']^{-1}$, it is defined over $K(x)$, and we have $(f) = n \cdot (\mathfrak{m}_1' - \mathfrak{m}_1)$. Furthermore, $\phi(x) = h(\mathfrak{m}_1 - \mathfrak{m}_0)$ is transformed into $\phi(x + s) = h(\mathfrak{m}_1' - \mathfrak{m}_0)$. Similarly, if we consider the generic specialization $(\mathfrak{m}_1, x) \to (\mathfrak{m}_n, n \cdot x)$ over K, we get $\phi(n \cdot x) = h(\mathfrak{m}_n - \mathfrak{m}_0)$. Now, $h(\mathfrak{m}_n - \mathfrak{m}_0)$ can be written as $h(n \cdot (\mathfrak{m}_1 - \mathfrak{m}_0)) \cdot h((F))$, and this is equal to $(\phi(x) \cdot F(\mathfrak{b}))^n$ by Lemma 4. Since $F(\mathfrak{b})$ is an element of $K(x)$, there exists a function ψ on J, which is defined over K, such that $\psi(x) = \phi(x) \cdot F(\mathfrak{b})$. Then, $\psi(x + s) \cdot \psi(x)^{-1}$ can be written as $(\phi(x + s) \cdot \phi(x)^{-1}) \cdot (F'(\mathfrak{b}) \cdot F(\mathfrak{b})^{-1})$, and this is equal to $h(\mathfrak{m}_1' - \mathfrak{m}_1) \cdot f(\mathfrak{b})^{-1}$. However, $\psi(x + s) \cdot \psi(x)^{-1}$ is nothing but $e_n(s.t)$, while $h(\mathfrak{m}_1' - \mathfrak{m}_1) \cdot f(\mathfrak{b})^{-1}$ is equal to $(\mathfrak{m}_1' - \mathfrak{m}_1, \mathfrak{b}; n) = (\mathfrak{a}, \mathfrak{b}; n)$. q. e. d.

With these preparations, the proof of the following theorem is a matter of technique:

THEOREM 4. *Let s and t be, respectively, invariant point and vanishing point of order n along I. Then, we have $e_n(s, t) = 1$.*

Proof. As in the proof of Theorem 3, we can find a positive divisor $\mathfrak{r}$ of C, which is rational over $k(u)$, such that its specialization $\mathfrak{r}'$ over the specialization $u \to a$ with reference to k is free from Q, and, moreover, such that the complete linear system $\mathfrak{L}_m - \mathfrak{r}$ is of degree nd. Pick two positive divisors $\mathfrak{m}_0$ and $\mathfrak{n}_0$ of C such that $n \cdot \mathfrak{m}_0$ and $n \cdot \mathfrak{n}_0$ both belong to $\mathfrak{L}_m - \mathfrak{r}$. We can assume that $\mathfrak{m}_0$ and $\mathfrak{n}_0$ are algebraic over $k(u)$, and such that the specializations $\mathfrak{m}_0'$ and $\mathfrak{n}_0'$ of $I(\mathfrak{m}_0)$ and $I(\mathfrak{n}_0)$ at the center of $k[[u - a]]$ have no point in common with $\mathfrak{r}' + Q$, and have no point in common with each other. Pick another two positive divisors $\mathfrak{m}$ and $\mathfrak{n}$ of C of degree d such that $S[\phi(\mathfrak{m} - \mathfrak{m}_0)] = s$ and $S[\phi(\mathfrak{n} - \mathfrak{n}_0)] = t$. Again, we can assume that $\mathfrak{m}$ and $\mathfrak{n}$ are algebraic over $k(u)$, and such that the specializations $\mathfrak{m}'$ and $\mathfrak{n}'$ of $I(\mathfrak{m})$ and $I(\mathfrak{n})$ at the center of $k[[u - a]]$ have no point in common with $\mathfrak{m}_0' + \mathfrak{n}_0' + \mathfrak{r}' + Q$, and have no point in common with each other. Finally, we can find four hypersurfaces F_0, F, H_0 and H in the ambient space of C, all algebraic over $k(u)$, such that they intersect C, respectively, at $n \cdot \mathfrak{m}_0 + \mathfrak{r}$, $n \cdot \mathfrak{m} + \mathfrak{r}$, $n \cdot \mathfrak{n}_0 + \mathfrak{r}$ and $n \cdot \mathfrak{n} + \mathfrak{r}$. We can assume that the specializations F_0', F', H_0' and H' of $I(F_0)$, $I(F)$, $I(H_0)$ and $I(H)$

at the center of $k[[u-a]]$ do not contain C'. Let $F_0(X)$, $F(X)$, $H_0(X)$ and $H(X)$ be homogeneous polynomials of order m, respectively associated with F_0, F, H_0 and H; similarly for $F_0'(X)$, $F'(X)$, $H_0'(X)$ and $H'(X)$. Then, the ratios $F(X)\cdot F_0(X)^{-1}$ and $H(X)\cdot H_0(X)^{-1}$ define numerical functions f and h on C. Similarly, the ratios $F'(X)\cdot F_0'(X)^{-1}$ and $H'(X)\cdot H_0'(X)^{-1}$ define functions f' and h' on C'. Here, as we can see, we have $(f)=n\cdot(\mathfrak{m}-\mathfrak{m}_0)$, $(h)=n\cdot(\mathfrak{n}-\mathfrak{n}_0)$, $(f')=n\cdot(\mathfrak{m}'-\mathfrak{m}_0')$ and $(h')=n\cdot(\mathfrak{n}'-\mathfrak{n}_0')$. Also, by the construction (f') and (h') are the divisors of f' and h' in the sense of valuation theory. Now, by the previous proposition $h(\mathfrak{m}-\mathfrak{m}_0)\cdot f(\mathfrak{n}-\mathfrak{n}_0)^{-1}$ is equal to $e_n(s,t)$. However, $h'(\mathfrak{m}'-\mathfrak{m}_0')$ and $f'(\mathfrak{n}'-\mathfrak{n}_0')$ are the unique specialization of $h(\mathfrak{m}-\mathfrak{m}_0)$ and $f(\mathfrak{n}-\mathfrak{n}_0)$ over the specialization

$$(u, F_0, F, H_0, H)\to(a, F_0', F', H_0', H')$$

with reference to k. Since $e_n(s,t)$ is an n-th root of unity, it is invariant under any specialization over k. Therefore, we get

$$e_n(s,t)=h'(\mathfrak{m}'-\mathfrak{m}_0')\cdot f'(\mathfrak{n}'-\mathfrak{n}_0')^{-1}.$$

Since t is a vanishing point along I, there exists a function h'' on C' such that $\mathfrak{n}'-\mathfrak{n}_0'$ is the divisor of h'' in the sense of valuation theory. This does not mean, of course, that h'' is defined by a ratio of homogeneous polynomials which do not vanish at Q. Any way, we get $h'=c\cdot(h'')^n$ with a certain constant c, and hence

$$h'(\mathfrak{m}'-\mathfrak{m}_0')\cdot f'(\mathfrak{n}'-\mathfrak{n}_0')^{-1}=h''(n\cdot(\mathfrak{m}'-\mathfrak{m}_0'))\cdot f'(\mathfrak{n}'-\mathfrak{n}_0')^{-1},$$

and this is equal to 1 by Lemma 4. q. e. d.

We recall, here, that $e_n(s,t)$ is a skew-symmetric bilinear form on the group of points of order n on J [6, p. 153]. The Theorem 4 asserts that the groups of invariant points and vanishing points of order n along I are the groups of annihilators of each other. In particular, each one of them determines the other uniquely.

5. Passage to inductive limits. We shall now consider the limit $n\to\infty$. Here, n runs over positive integers which are not divisible by the characteristic p, and which "finally" contain powers of any prime l different from p. More conveniently, we consider the limits $l^\nu\to\infty$ for all l different from p. We fix one such l in the following.

Let $\mathfrak{w}_\nu$ be the additive group of points of J of order l^ν, and let $\mathfrak{w}$ be the union of $\mathfrak{w}_0, \mathfrak{w}_1, \cdots$. If we adjoin all elements of $\mathfrak{w}$ to $k(u)$, we get an algebraic extension K of $k(u)$. Similarly, an algebraic extension $k((u-a))(I(\mathfrak{w}))$ of $k((u-a))$ is defined, and it is the compositum of $I(K)$ and $k((u-a))$. Let $\mathfrak{G}$ and $\bar{\mathfrak{g}}$ be the Galois groups of K over $k(u)$

and $k((u-a))(I(\mathfrak{w}))$ over $k((u-a))$, respectively, with Krull's topologies. If $\bar{\sigma}$ is in $\bar{\mathfrak{g}}$, then $\sigma = I^{-1}\cdot\bar{\sigma}\cdot I$ is in $\mathfrak{G}$. Moreover, the correspondence $\bar{\sigma}\to\sigma$ is a continuous isomorphism of $\bar{\mathfrak{g}}$ into $\mathfrak{G}$, and we get a compact image group $\mathfrak{g}$. The group $\mathfrak{g}$ is generally called the *decomposition group* of K over $k(u)$ along I. Here, since k is algebraically closed, $\mathfrak{g}$ coincides with the *inertia group* of K over $k(u)$ along I. Since the group law in J is defined over $k(u)$, the group $\mathfrak{G}$, hence in particular the group $\mathfrak{g}$, operates on $\mathfrak{w}$. On the other hand, if we denote by Z_l the ring of l-adic integers, and by Q_l its quotient-field, $\mathfrak{w}$ is isomorphic to a $2g$-fold direct product of the additive group Q_l modulo 1. This isomorphism is called an *l-adic coordinate system* in the group $\mathfrak{w}$. With reference to a fixed l-adic coordinate system, $\mathfrak{G}$ is represented by a group of matrices of degree $2g$ with coefficients in Z_l. We call this matric group the *abstract monodromy group of the fibre system $\{u\times J\}$ at l*, while the *abstract local monodromy group* of the fibre system along I at l is defined by using $\mathfrak{g}$ instead of $\mathfrak{G}$. At present, we are interested in the structure of abstract local monodromy groups. We shall prove the following theorem:

THEOREM 5. *We can choose an l-adic coordinate system in $\mathfrak{w}$ such that any element σ of the inertia group $\mathfrak{g}$ is represented by a matrix of the form*[5]

$$\begin{bmatrix} 1 & c & & \\ 0 & 1 & & \\ & & 1_{2g-2} & \end{bmatrix}.$$

Here, the correspondence $\sigma\to c$ is a continuous isomorphism of $\mathfrak{g}$ into the additive group Z_l.

Proof. We fix, once for all, an isomorphism 1_g of the multiplicative group of l^ν-th roots of unity for $\nu=0,1,\cdots$ onto the additive group Q_l modulo 1. Also, let $\mathfrak{v}_\nu$ be the group of vanishing points of order l^ν along I, and let $\mathfrak{v}$ be the union of $\mathfrak{v}_0,\mathfrak{v}_1,\cdots$. Then, we can find an isomorphism i of $\mathfrak{v}$ onto Q_l modulo 1. According to Lemma 2, we can extend the isomorphism i to an isomorphism of the group of invariant points of order l^ν for $\nu=0,1,\cdots$ along I onto a $(2g-1)$-fold direct product of Q_l modulo 1. Here, we agree to take the image of $\mathfrak{v}$ as the first factor of the product. Also, we keep the notation i for the extension. Now, let ν be an arbitrary non-negative integer, and let t_ν be an element of $\mathfrak{v}_\nu$ such that $i(t_\nu)\equiv(l^{-\nu}0\cdots 0)$ mod. 1. Then, we have $l\cdot t_{\nu+1}=t_\nu$ for $\nu=0,1,\cdots$. If we denote $e_n(*,*)$,

[5] The symbol 1_{2g-2} denotes a unit matrix of degree $2g-2$: two blanks are zero matrices.

temporarily, by $e_\nu(*,*)$ for $n=l^\nu$, we can find an element, say s_ν, of $\mathfrak{w}_\nu$ such that $1g(e_\nu(s_\nu,t_\nu))\equiv l^{-\nu} \bmod 1$. If we impose the further condition that $e_\nu(s_\nu,s)=1$ for all invariant points s of order l^ν along I satisfying $i(s)\equiv(0*\cdots*) \bmod 1$, then, s_ν is unique modulo $\mathfrak{v}_\nu$. In fact, if s_ν' is another element with similar properties, $s_\nu'-s_\nu$ is annihilated by all invariant points of order l^ν along I. Hence, by Theorem 4 this is an element of $\mathfrak{v}_\nu$. Since the converse is also true, the number or distinct s_ν is equal to l^ν. Since $l\cdot s_{\nu+1}$ is one of the s_ν, we can find a sequence $s_0, s_1, \cdots$ satisfying $l\cdot s_{\nu+1}=s_\nu$ for $\nu=0,1,\cdots$. Now, let s be an arbitrary element of $\mathfrak{w}$, say in $\mathfrak{w}_\nu$; put $1g(e_\nu(s,t_\nu))\equiv n\cdot l^{-\nu} \bmod 1$ with some integer n. Then, by the same reason as above, $s-n\cdot s_\nu$ is an invariant point of order l^ν along I. Hence, we can extend the isomorphism i to an l-adic coordinate system θ in $\mathfrak{w}$ by

$$\theta(s)\equiv(n\cdot l^{-\nu}\, i(s-n\cdot s_\nu)) \bmod 1.$$

In fact, $\theta(s)$ is independent of the choice of ν, and θ gives an isomorphism of $\mathfrak{w}$ onto the $2g$-fold direct product of Q_l modulo 1. Finally, let σ be an arbitrary element of $\mathfrak{g}$, and let ν be a non-negative integer. Then, by Theorem 3 we can find an integer c_ν such that $\sigma(s_\nu)=s_\nu+c_\nu\cdot t_\nu$. Since we have $l\cdot s_{\nu+1}=s_\nu$ and $l\cdot t_{\nu+1}=t_\nu$, we have $c_{\nu+1}\cdot t_\nu=c_\nu\cdot t_\nu$, i.e., $c_{\nu+1}\equiv c_\nu \bmod l^\nu$. Therefore, the sequence $c_0, c_1, \cdots$ has a limit, say c, in Z_l such that $c\equiv c_\nu \bmod l^\nu$ for $\nu=0,1,\cdots$. On the other hand, since s_ν generates $\mathfrak{w}_\nu$ modulo invariant points of order l^ν along I, we conclude that $k((u-a))(I(\mathfrak{w}_\nu))$ coincides with $k((u-a))(I(s_\nu))$. Therefore, the correspondence $\sigma\to c$ gives a continuous isomorphism of $\mathfrak{g}$ into Z_l. Let $M(\sigma)$ be the matrix in the theorem with the above defined c as its $(1,2)$-coefficient. Then, for any element s in $\mathfrak{w}$, we have $\theta(\sigma(s))\equiv\theta(s)\cdot M(\sigma) \bmod 1$. q.e.d.

As we know in general, there exists a skew-symmetric matrix E of degree $2g$ with coefficients in Z_l such that $1g(e_\nu(s,t))\equiv l^\nu\cdot\theta(s)\cdot E\cdot{}^t\theta(t) \bmod 1$ for s and t in $\mathfrak{w}_\nu$ for $\nu=0,1,\cdots$. We know also that E^{-1} exists, and has coefficients in Z_l [6, p. 156]. Now, in the present case, the proof of Theorem 5 shows that E has the following special form:

$$E=\begin{bmatrix} \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} & & \\ & \begin{vmatrix} & \\ & E' \end{vmatrix} & \end{bmatrix}.$$

Here, of course, E' is a similar matrix as E.

We shall discuss some supplements of Theorem 5. We first note that the image of $\mathfrak{g}$ in Z_l is compact, hence it is a closed additive subgroup of Z_l.

However, any closed additive subgroup of Z_l is an ideal of the ring Z_l. Therefore, the image in question is of the form $l^\rho \cdot Z_l$ with $0 \leqq \rho \leqq \infty$. Here, we consider l^ρ as 0 for $\rho = \infty$. We shall show that ρ is an *invariant of the fibre system* in the following sense:

SUPPLEMENT 1. *The number ρ depends neither on the isomorphism I nor on the l-adic coordinate system θ.*

In fact, let θ' be a coordinate system in $\mathfrak{w}$ which gives rise to a similar representation of $\mathfrak{g}$ as the one stated in Theorem 5. Let σ be an element of $\mathfrak{g}$, and let $M'(\sigma)$ be the corresponding matrix with respect to the new system. Then, we can find a non-singular matrix L of degree $2g$ such that $M'(\sigma) = L^{-1} \cdot M(\sigma) \cdot L$. Here, both L and L^{-1} have their coefficients in Z_l. Therefore, if c and c' are the $(1,2)$-coefficients of $M(\sigma)$ and $M'(\sigma)$, respectively, we can verify that c and c' differ only by a unit of Z_l. Also the change of I amounts to replace $\mathfrak{g}$ by a conjugate in $\mathfrak{G}$, and the argument is similar.

We note that the invariant ρ does not depend, by Lemma 1, on the choice of the group law of J. Now, in the case of $p = 0$, we can show that $\rho = 0$ for all prime l (cf. [3]). In the modular case, we are satisfied, temporarily, with stating the following remark:

SUPPLEMENT 2. *There exists a linear pencil on the surface V such that the invariant ρ of the associated fibre system of Jacobian varieties along any degenerate fibre is always finite, and also such that $\rho = 0$ for almost all prime l.*

Since our proof of this supplement is not local, we shall reserve it for our global theory.

THE JOHNS HOPKINS UNIVERSITY.

REFERENCES.

[1] W. L. Chow, "The Jacobian variety of an algebraic curve," *American Journal of Mathematics*, vol. 76 (1954), pp. 453-476.
[2] J. Igusa, "Fibre systems of Jacobian varieties," *ibid.*, vol. 78 (1956), pp. 171-199.
[3] ———, "Monodromy groups connected with algebraic surfaces," *to appear*.
[4] M. Rosenlicht, "Generalized Jacobian varieties," *Annals of Mathematics*, vol. 59 (1954), pp. 505-530.
[5] A. Weil, *Foundations of algebraic geometry*, New York (1946).
[6] ———, *Variétés Abéliennes et courbes algébriques*, Paris (1948).
[7] O. Zariski, *Algebraic surfaces*, New York (1948).
[8] ———, "Complete linear systems on normal varieties and a generalization of a lemma of Enriques-Severi," *Annals of Mathematics*, vol. 55 (1952), pp. 552-592.

SIMULTANEOUS RESOLUTION FOR ALGEBRAIC SURFACES.*

By SHREERAM ABHYANKAR.

IN MEMORY OF MY LATE FRIEND PROFESSOR IRVIN S. COHEN (1917-1955).

---

1. Introduction. Now that one has the resolution theorem for algebraic surfaces[1] one may naturally ask the following question concerning *simultaneous resolvability*: Given a two dimensional algebraic function field K and a finite algebraic extension K^* of K does there exist a non-singular projective model of K whose K^*-normalization is also non-singular? This and related weaker (global as well as local) questions were raised by us in part B of Section 6 of [4]. The purpose of the present paper is to answer some of these questions. The main results of this paper are equally novel for fields of zero characteristic (in particular for the complex ground field) or for modular fields. From a pair of projective normal models V and V^* respectively of K and K^* such that the transformation between V and V^* is free from fundamental points on both (i.e. V^* is a K^*-normalization of V), instead of requiring that both be non-singular one may require that at least one of them be non-singular. Since requiring V to be non-singular is simply equivalent to the resolution theorem for K it is of no added content; however requiring V^* to be non-singular asks much more about the pair (K, K^*) than the mere resolution theorem for K^* and hence we could call this the question of *weak simultaneous resolvability* for the pair (K, K^*). If we restrict our attention to a zero dimensional valuation of K^* and its K-restriction then we get the problems of *local simultaneous resolution* and *local weak simultaneous resolution* respectively. The results of this paper are quite in conformity with the conjectured statements concerning these questions made in Section 6 of [4]. It is obvious that these simultaneous resolvability questions are related to the problem: Given an involution T on a variety V to remove the fundamental points of these involutions (i.e. the points on V which are fundamental for the rational map of V onto the Chow variety of the involution T). We intend to study this involution problem at some future opportunity.

---

* Received March 15, 1956.

[1] For characteristic zero see Zariski [8], for non-zero characteristic with algebraically closed ground fieelds see Abhyankar [3] and for generalization to perfect ground fields see Abhyankar [4].

Now to describe the results of this paper in greater detail, let K be a two dimensional algebraic function field over ground field k of characteristic p and let K^* be a finite algebraic extension of K. Let v^* be a zero dimensional valuation of K^* and let v be the K-restriction of v^*.

In Section 2 we settle the question of local weak simultaneous resolution in the affirmative; i.e., we prove that if v^* can be uniformized (this condition is certainly satisfied if k is perfect, see [4]), then v^* can be uniformized on a K^*-normalization of a projective normal model of K. This result will then eliminate the use of the Zariski factorization theorem for an antiregular transformation between two surfaces into local quadratic transformation (see Theorem 3 of [2]) from our proof of the local uniformization for algebraic surfaces over algebraically closed modular ground fields [3] and its generalization to perfect ground fields [4]; more explicit references to these points will be made in Section 2. Since generalization of such a precise factorization theorem for higher dimensional varieties is not possible, this elimination of the factorization theorem was essential before attempting uniformization for higher varieties (over modular ground fields). In Section 3 we prove that if k is algebraically closed, $p=0$ and v is rational, then local simultaneous resolution holds, i.e., there exists a projective normal model V of K on which v has a simple center and such that on the K^*-normalization of V the center of v^* is simple. In Section 4 we give an affirmative answer to the (global) weak simultaneous resolvability question in case k is algebraically closed, $p=0$ and K^*/K is galois.

The more novel part of the paper (Sections 7 and 8) dealing with (global) simultaneous resolution is preceded by a few words on the branch locus (Section 5) and some theorems on kummer extensions of unique factorization domains (Section 6). In Section 7, assuming k to be perfect, we prove that either if K^*/K is a quadratic extension and $p \neq 2$ or if K^*/K is a cubic cyclic extension, $p \neq 3$ and k contains primitive cube roots of unity then there exists a *non-singular* projective model V of K whose K^*-normalization is also *non-singular* and the branch locus on V is a non-singular (in general reducible) curve. It is expected that this result should be helpful towards a comparison of the various structural properties and associated invariants of the fields K and K^*. In Section 8, we prove that the answer to the simultaneous resolution question, although positive for quadratic and cubic cyclic extensions, is negative in general. More precisely we prove that if K possesses minimal models (i.e., if K is not the function field of a ruled surface) and if q is an arbitrary prime number with $q \neq p$ and $q > 3$ then there exist (lots of) q-cyclic extensions K' of K such that for any non-singular projective model of K the K'-normalization is necessarily singular.

Finally in Section 9 we give partial generalizations to higher dimensional varieties of the results of Sections 2, 3 and 7. These considerations might throw some light on Zariski's proposed method of resolution of singularities for higher varieties [13].

The notations in this paper will be the same as those used in [1], [3] and [4].

2. Uniformization on a derived normal model. In our proof of the local uniformization theorem for algebraic surfaces over algebraically closed modular ground fields ([3], see proof of Theorem 3 in Section 6 on "Cyclic extensions of prime degree $q \neq p$; coming down") and also in its generalization to perfect ground fields ([4], see Theorem 3 of Section 2) we had to deal with the following situation: Given a galois extension K^* of a two dimensional algebraic function field K and a zero dimensional valuation v of K having a unique extension v^* to K^*, knowing that v^* can be uniformized the problem was to uniformize v^* on a K^*-normalization of some projective normal model of K. This was achieved by using the Zariski factorization theorem of antiregular transformations between algebraic surfaces into local quadratic transformations (Theorem 3 of [2]). A second instance where we faced a similar situation in our uniformization proof was as follows (Theorem 1 of Section 4 of [3] and Theorem 2 of Section 2 of [4]): K^* is a finite separable algebraic extension of a two dimensional algebraic function field K, v^* a zero dimensional valuation of K^* and v the K-restriction of v^*, such that for a projective normal model V of K and its K^*-normalization V^* we have $d(R^*:R) = 1$ where R^* and R are the quotient rings of the centers of v^* and v respectively on V^* and V. Given that v^* can be uniformized the problem was to uniformize v^* on a K^*-normalization of some projective normal model of K which dominates V.

Both of these are obviously special cases of the weak local simultaneous resolution theorem for algebraic surfaces, i. e., Theorem 1 below. The proof of Theorem 1 in case v is of rational rank greater than one (this is the easy case) is essentially the same as of Theorem 2 of [4] while for the case when v is rational we have to use one new trick. Consequently the only deep property of quadratic transformations of two dimensional regular local domains which is used in the proof of Theorem 1 is Theorem 2 of [2] (which may be called weak local uniformization theorem). Since both the instances of the use of the Zariski factorization theorem in our proof of the local uniformization theorem for algebraic surfaces are special cases of Theorem 1 below, we have now eliminated the use of the factorization theorem. This

is important for the uniformization problem for higher varieties (over modular ground fields), since to generalize the factorization theorem, in any form, to higher varieties is a very difficult problem at all events (even in the classical case). On the other hand the weak local uniformization theorem is true for h gher varieties at least over algebraically closed ground fields of characteristic zero (see Section 9). Besides these applications the following theorem on local weak simultaneous is of importance in itself.

THEOREM 1. (*Local weak simultaneous resolution*). *Let K/k be a two dimensional algebraic function field and let K^* be a finite algebraic extension of K. Let v^* be a zero dimensional valuation of K^*/k and let v be the K-restriction of v^*. Assume that v^* can be uniformized. Then v^* can be uniformized on the K^*-normalization of a projective normal model of K/k.* [More precisely if R^* is the quotient ring of a simple center on some projective normal model of K/k then we can find a quadratic transform of R^* along v^* which is a quotient ring on a K^*-normalization of a projective normal model of K/k.]

Proof. (In general we shall follow the proof of Theorem 2 of [4] or rather we shall sharpen that proof). Let (R^*, M^*) be the quotient ring of a simple center of v^* on some projective normal model of K^*/k. Let $S = R^* \cap K$, $Q = M^* \cap K$ and $P = QR^*$. Now if e is an element of K which is integral over S then e is integral over R^*, i.e. e is in R^* and hence e is in $R^* \cap K = S$. Therefore S is integrally closed in K. Hence $K =$ the quotient field of S if and only if tr. deg. (transcendence degree of) $S/k =$ tr. deg. K/k. We proceed to arrange matters so that tr. deg. S/k $=$ tr. deg. K/k. Let q_1, q_2 be a transcendence basis of K/k. Replacing q_i by $1/q_i$ if necessary, we may assume that $q_1, q_2 \in R_v$. Replacing R^* by a quadratic transform (see Definition 3 of [2]) along v^* we may assume that $q_1, q_2 \in R^*$ (Lemma 12 of [2]), i.e. that K is the quotient field of S.

Now fix a non-zero element z in M_v. Again replacing R^* by a quadratic transform along v^* we may assume that (Theorem 2 of [2]):

$$z = x^a y^b d$$

where (x, y) is a basis of M^*, a and b are non-negative integers, d is a unit in R^* and where we have either that $b = 0$ and $a > 0$ or that $v^*(x)$ and $v^*(y)$ are rationally independent. We proceed to show that in either case P is primary for M^*; and that will then, in view of Proposition 1 of [4], complete the proof.

Case 1, $a > 0$ *and* $b = 0$. Now K is the quotient field of $S = R^* \cap K$

and y is algebraic over K. Hence for some non-zero element e of S, ey is integral over S. Let $(ey)^n + f_{n-1}(ey)^{n-1} + \cdots + f_1(ey) + f_0 = 0$ be the minimal equation of ey over K. Then $f_0, f_1, \cdots, f_n$ belong to S where we take $f_n = 1$. Let $g_i = f_i e^i$. Then $g_0, g_1, \cdots, g_n$ belong to S and we have $g_n y^n + g_{n-1}y^{n-1} + \cdots + g_0 = 0$. The minimality of the equation implies that $g_0 \neq 0$. Thus $g_0 = -y(g_n y^{n-1} + g_{n-1}y^{n-1} + \cdots + g_1)$ is a non-zero element of $S = R^* \cap K$ which is divisible by y in R^*. Let $g_0 = x^u h^*$ where u is a non-negative integer and h^* is an element of R^* not divisible by x. Let $h = g_0^a / z^u$. Then h is a non-zero element of $R^* \cap K = S$ which is not divisible by x in R^* but is divisible by y in R^*. Since h is in $yR^* \subset M^*$ and h is in K, we have that h is in $M^* \cap K = Q$. Suppose if possible that P is of rank one. Then $P \subset tR^*$ where t is a non-zero non-unit in R^* (since every rank one prime ideal in R^* is principal). Now $z = x^a d \in P \subset tR^*$ and hence $tR^* = x^{a^*}R^*$ where a^* is a positive integer $(a^* \leqq a)$. But $h \in P \subset tR^* = x^{a^*}R^*$ is a non-unit in R^* which is not divisible by x. This is a contradiction. Therefore P is of rank greater than one, i.e., rank $P = 2$ and hence P is primary for M^*.

Case 2, $a > 0$ and $b > 0$. The proof is the same as in Theorem 2 of [4.]

Remark 1. It is expected that the above proof can be easily adapted to the case of *absolute surfaces* [2].

Remark 2. In the above proof we had to arrange matters so that tr. deg. $S/k =$ tr deg. K/k. For a normal integral domain A^* having for quotient field an algebraic function field K^*/k and for a field K containing k such that K^* is a finite algebraic extension of K, *a priori* we may have tr. deg. $(A^* \cap K)/k <$ tr. deg. K/k. This can be illustrated by the following example: Let k be an algebraically closed field of characteristic p, $K^* = k(x)$ where x is a transcendental, $K = k(x^n)$ with $n > 1$ (and $n \not\equiv 0(p)$ if $p \neq 0$), $A^* = k[1/(x-1)]$. Then $A^* \cap K = k$.

Proof. Let v be the valuation of K/k given by $v(x^n - 1) = 1$. Let v^* be the valuation of K^*/k given by $v^*(x-1) = 1$. Then v^* is an extension of v and v has at least one more extension v' to K^*. Now suppose, if possible, that there exists y in $A^* \cap K$ which is not in k. Then $y \notin k$ implies that there exists a valuation w^* of K^*/k such that $w^*(y) < 0$. Since $y \in A^*$ we must have $v^* = w^*$. Since $y \in K$, $v^*(y) < 0$ implies that $v(y) < 0$ and hence $v'(y) < 0$. Therefore $v' = w^* = v^*$ which is a contradiction. (This construction is suggested by Zariski's paper on Hilbert's 14th problem.)

3. Simultaneous uniformization of rational valuations. The aim of this section is to prove

THEOREM 2. (*Local simultaneous resolution for rational valuations*). *Let K be a two dimensional algebraic function field over an algebraically closed ground field k of characteristic zero and let v be a zero dimensional rational valuation of K/k. Let K^* be a finite algebraic extension of K and let v^* be an extension of v to K^*. Then there exists a projective normal* (in fact non-singular) *model V of K/k on which the center of v is at a simple point and such that the center of v^* on a K^*-normalization of V is also a simple point.*

Before proving this theorem we must make the ramification theoretic preparation for it. The following proposition on rational groups was conjectured by us and a proof of it was kindly provided by Professor Iwasawa.

PROPOSITION 1. *Let G be a torsion free abelian group of rational rank one and let H be a subgroup of G of finite index. Then G/H is cyclic.*

Proof. We may arrange matters so that G is a subgroup of the additive group R of rational numbers and so that the subgroup Z of integers is contained in H. Let $R^* = R/Z$, $G^* = G/Z$ and $H^* = H/Z$. Then G/H is isomorphic to G^*/H^* and hence it is enough to show that G^*/H^* is cyclic. Now the elements of R^* can be considered to be rational numbers r with $0 \leqq r < 1$, and the addition in R^* then being addition modulo 1. Given integers r and s with $0 \leqq r < s$, we write $s = \prod_{i=1}^{n} p_i^{u_i}$, where $p_1, p_2, \cdots, p_n$ are distinct prime numbers and $u_1, u_2, \cdots, u_n$ are positive integers. There exist, by partial fraction theory, unique integers r_i with $0 \leqq r_i < p_i^{u_i}$ such that $r/s = \sum_{i=1}^{n} r_i/p_i^{u_i}$. Thus R^* has the direct sum representation: $R^* = \oplus_{p_i \in P} S(p_i)$ where P is the set of all prime numbers and

$$S(p_i) = S_\infty(p_i) = \bigcup_{j=1}^{\infty} S_j(p_i)$$

where $S_j(p_i)$ is the subgroup of elements t of R^* for which $p_i^j t$ is an integer. Now let t be any given non-zero element of G^*. We can write $t = r/s$ where r and s are coprime integers such that $0 < r < s$. Let $s = \prod_{i=1}^{n} p_i^{u_i}$ where $p_1, p_2, \cdots, p_n$ are distinct prime numbers and $u_1, u_2, \cdots, u_n$ are positive integers. Then there exist unique integers r_i with $0 < r_i < p_i^{u_i}$ such that

$t = \sum_{i=1}^{n} r_i/p_i^{u_i}$. If $n = 1$ then obviously $t \in G^* \cap S(p_1)$. Now suppose $n > 1$. Then we can find integers a and b such that

$$bp_1^{u_1} + a\prod_{i=2}^{n} p_i^{u_i} = 1; \text{ so that } (a\prod_{i=2}^{n} p_i^{u_i})r_1 = r_1 + cp_1^{u_1},$$

where c is the integer $(-br_1)$. Hence

$$(a\prod_{i=2}^{n} p_i^{u_i})t = r_1p_1^{-u_1} + c + \sum_{i=2}^{n} ar_id_i,$$

where d_i is the integer $(\prod_{j=2}^{n} p_j^{u_j})p_i^{-u_i}$ so that $r_1p_1^{-u_1} \in G^* \cap S(p_1)$. Similarly $r_ip_i^{-u_i} \in G^* \cap S(p_i)$ for $i = 2, 3, \cdots, n$. Therefore $G^* = \oplus_{p_i \in P} G^* \cap S(p_i)$ and similarly $H^* = \oplus_{p_i \in P} H^* \cap S(p_i)$. Now $G^* \cap S(p_i) = S_{v_i}(p_i)$ where v_i is either a non-negative integer or the symbol ∞ and similarly $H^* \cap S(p_i) = S_{w_i}(p_i)$. We have that $w_i \leqq v_i$ and $G^*/H^* = \oplus_{p_i \in P}[S_{v_i}(p_i)/S_{w_i}(p_i)]$. Since G^*/H^* is finite we must have $v_i = w_i$ except for a finite number of subscripts $e_1, e_2, \cdots, e_m$ for which $v_{e_q} < w_{e_q} < \infty$ (for $q = 1, 2, \cdots, m$). Since $S_{v_i}(p_i)/S_{w_i}(p_i)$ is cyclic of order $p_i^{v_i - w_i}$ for $i = e_1, e_2, \cdots, e_m$; we conclude that G^*/H^* is cyclic of order $\prod_{j=e_1, e_2, \cdots, e_m} p_j^{(v_j - w_j)}$.

PROPOSITION 2. *Let K be an algebraic function field over an algebraically closed ground field k of characteristic zero and let K^* be a galois extension of K. Let v^* be a zero dimensional rational valuation of K^*/k and let v be the K-restriction of v^*. Then the splitting group of v^* over v is cyclic.*

Proof. The proof follows from Proposition 1 above and Satz 3 of Krull [5], in view of the fact that the residue field of v is the algebraically closed field k of characteristic zero.

PROPOSITION 3. *Let K be a two dimensional algebraic function field over an algebraically closed ground field k and let K^* be a galois extension of K. Let (R, M) be the quotient ring of a simple point on some projective model of K/k. Let v be a real zero dimensional valuation of K/k having center M in R and let v^* be an extension of v to K^*. Let R_n be the n-th quadratic transform* (Definition 3 of [2]) *of R along v and let (R_n^*, M_n^*) be the local ring in K^* lying above R_n such that v^* has center M_n^* in R_n^*. Then there exists an integer m such that for all $n \geqq m$ the splitting field of R_n^* over R_n is the same as the splitting field of v^* over v.*

The proof of this proposition is contained in the proof of Proposition 4

of Section 4 of [3]. That proof was based among other things on the fact that $\bigcup_{n=1}^{\infty} R_n = R_v$ (Lemma 10, Section 3 of [3]). Another proof based on the weak local uniformization theorem (Proposition 3, Section 3 of [3]) will be given Section 9 (Proposition 3A) and this proof will be applicable to higher dimensional varieties over ground fields of characteristic zero which are algebraically closed. In [3], the weak local uniformization was deduced from the theorem: $\bigcup_{n=1}^{\infty} R_n = R_v$. However, in Section 9 the weak uniformization theorem will be proved for higher varieties (real valuations, characteristic zero) and in particular for surfaces without the use of the theorem: $\bigcup_{n=1}^{\infty} R_n = R_v$. This last theorem has no known generalization to higher varieties.

Proof of Theorem 2. (We shall freely use the results of Section 2 of [3] without making specific references.) By the local uniformization theorem, we can find a projective normal model of K/k on which the center of v is a simple point. Let (R, M) be the quotient ring of this simple point. Let K' be a galois extension of K containing K^* and let v' be an extension of v^* to K'. Let K_s' be the splitting field of v' over v and let K_s^* be the compositum of K^* and K_s'. By Proposition 2, K'/K_s' is cyclic and hence K_s^*/K_s' is also cyclic. Let (R', M') be the local ring in K' lying above R such that v' has center M' in R'. Let R_s', R_s^*, R^* and M_s', M_s^*, M^* be the respective intersections of R' and M' with K_s', K_s^* and K^*. By Proposition 3 we may assume that K_s' is the splitting field of R' over R. Then K_s^* is the splitting field of R' over R^*. Let (x, y) be a basis of M. Then (x, y) is a basis also of M_s'. We can find a primitive element z of K_s^*/K_s' with minimal polynomial $X^n - t$ where t is in R_s'. If $n = 1$ there would be nothing to prove; so assume that $n > 1$. By the weak uniformization theorem (Lemma 10 of [3]), after replacing R_s' by a quadratic transform (S, N) along the K_s'-restriction of v', we may arrange matters so that $t = f^u d$ where (f, g) is a basis of N, u is a non-negative integer and d is a unit in S. Since k is algebraically closed, and (x, y) is a basis of M_s' as well as of M, it is clear that after replacing R by its corresponding quadratic transform along v we may assume that $S = R_s'$.

Suppose, if possible, that u and n have a common factor $m > 1$. Since k is algebraically closed of characteristic zero, $X^n - f^u d$ factors in $k[[f, g]][X]$ into m pairwise coprime fatcors. Since $k[[f, g]]$ is the completion of R_s' we have that R_s' splits in K_s^* into more than one local ring; this is a contradiction. Therefore u is prime to n. Hence by the argument used in the proof of Theorem 2 of Section 5 of [3] we conclude that R_s^* is regular.

Since K_s^* is the splitting field of R' over R^*, the degree of R_s^* over R^* is one and hence R^* is also regular.

4. Resolution on a derived normal model. The purpose of this section is to prove

THEOREM 3. *Let K be a two dimensional algebraic function field over an algebraically closed ground field k of characteristic zero and let K^* be a galois extension of K. Then there exists a projective normal model of K/k whose K^*-normalization is non-singular.*

The proof of this theorem will consist of several remarks.

(1) (This remark is valid for an arbitrary finite algebraic extension K^* of any dimensional algebraic function field K over an arbitrary ground field k). Let V^* be a projective model of K^*/k and let T be the rational transformation from V^* onto the Chow variety V of sets of points of the involution on V^* defined by the subfield K of K^*. Let W^* be a normalization of V^* and let U be the rational transformation from W^* onto the Chow variety W of sets of points of the involution on W^* defined by the subfield K of K^*. Finally let W_1 be a normalization (in K) of W. Then we have that

(a) T has no fundamental point on V^* if and only if U has no fundamental points on W^*.

(b) U has no fundamental points on W^* if and only if W^* is a K^*-normaliaztion of some projective normal model of K/k. In fact if U has no fundamental points on W^* then W^* is a K^*-normalization of W_1.

(2) Let V be a projective normal model of K/k and let $Q = (P_1, P_2, \cdots, P_n)$ be a finite set of (distinct) points of V. Let $V^{(1)}$ be the surface obtained by applying a local quadratic transformation to V centered at P_1 and followed by normalization. Let $P_2^{(1)}$ be the point on $V^{(1)}$ which corresponds to P_2. Let $V^{(2)}$ be the surface obtained by applying a local quadratic transformation to $V^{(1)}$ centered at $P_2^{(1)}$ and followed by normalization. And so on until we obtain a normal model $V^{(n)}$ of K/k. It is clear $V^{(n)}$ depends (up to a well defined biregular transformation) only on the set Q and not on the order in which the points P_i are taken. We shall say that $V^{(n)}$ is obtained from V by applying a quadratic transformation centered at the set of points Q and followed by normalization.

(3) Let V^* be a K^*-normalization of some projective normal model of K/k. Then the singular points of V^* (which are finite in number) arrange

themselves in complete K-conjugate sets of points (since K^*/K is galois). Let $P_1, P_2, \cdots, P_n$ be such a complete K-conjugate set of singular points of V^* and let W^* be the surface obtained from V^* by applying a quadratic transformation centered at $(P_1, P_2, \cdots, P_n)$ and followed by normalization. Then it follows by (1) that W^* is K^*-normalization of a projective normal model of K/k.

(4) Let again V^* be a K^*-normalization of some projective normal model of K/k and let $P_1, P_2, \cdots, P_N$ be the set of all the singularities of V^*. Let V_1^* be the surface obtained from V^* by applying a quadratic transformation centered at $(P_1, P_2, \cdots, P_N)$ and followed by normalization. Then (2) and (3) tells us that V_1^* is a K^*-normalization of some projective normal model of K/k.

(5) Let the notation be as in (4). Let $g(P_i)$ be the character of P_i as defined in Part VI of Zariski [8]. Let $h = \max(g(P_1), g(P_2), \cdots, g(P_N))$. Let V_2^* be the surface obtained from V_1^* as V_1^* was obtained from V^*. So on until we have obtained a projective normal model V_h^* of K^*/k which [by (4)] is a K^*-normalization of some projective normal model of K/k. In view of (2), it follows from the proof given by Zariski in Part VI of [8] that V_h^* is non-singular.

5. Preliminaries on the branch locus. In Lemmas 1 and 2 of [1] we had collected well known facts concerning the notion of derived normal model of an algebraic variety in a finite algebraic extension of its function field due to Zariski; there we unnecessarily restricted the extension to be separable and the ground field to be algebraically closed. For completeness we restate

LEMMA 1. *Let K be an r-dimensional algebraic function field and K^* a finite algebraic extension of K. Let V and V^* be normal projective models respectively of K and K^* such that the rational transformation from V^* onto V and its inverse map T^{-1} are both free from fundamental points. For a fixed irreducible subvariety W of V let $W_1^*, W_2^*, \cdots, W_s^*$ be the irreducible subvarieties of V^* corresponding to W. Let R and R_i^* be the quotient rings respectively of W and W_i^* on V and V^*. Then $R_1^*, R_2^*, \cdots, R_s^*$ are exactly the local rings in K^* lying above R.*

Proof. Let A be an affine coordinate ring of V contained in R. Let V' be a projective model of K^* having the integral closure A^* of A in K^* as an affine coordinate ring. Then to W there correspond on V' a finite

number of irreducible subvarieties $W_1', W_2', \cdots, W_t'$; V' is normal at $W_1', W_2', \cdots, W_t'$ and their quotient rings on V' are exactly the local rings in K^* lying above R. Furthermore it is clear that the birational transformation from V' *to* V^* is biregular at $W_1', W_2', \cdots, W_t'$ and maps these onto $W_1^*, W_2^*, \cdots, W_s^*$. Hence $s = t$ and after a suitable rearrangement we have $R_i = Q(W_i^*, V^*) = Q(W_i', V')$ for $i = 1, 2, \cdots, t$.

LEMMA 2. *Given K, K^* and V as in Lemma* 1, *V^* exists as in Lemma* 1 *and is unique up to a biregular transformation.*

Proof. Unicity follows from Lemma 1. Existence is proved on pages 68-70 of [12] or also in [6].

We call V^* a derived normal model of V in K^* or a K^*-normalization of V. By the *branch locus* $B(T)$ on V of the rational transformation T from V^* onto V we shall mean the set of all (*algebraic*) points P of V such that P is on some irreducible subvariety of V which is a branch subvariety for T.

LEMMA 3. *An irreducible subvariety W of V is a branch subvariety for T if and only if $W \subset B(T)$. Furthermore, $B(T) = V$ or $B(T)$ is a proper subvariety of V according as K^*/K is inseparable or separable.*

Proof. The proof is the same as that of Lemma 3 of [1] except that in the alternative proof given there in terms of symmetric product, any reference to "rational points" should be replaced by "algebraic points."

Remark 3. Let K/k be an r-dimensional algebraic function field and K^* a finite separable algebraic extension of K. Let V be a non-singular projective model of K/k and let V^* be K^*-normalization of V. Let D be the branch locus on V. If k is algebraically closed, then Zariski's theorem (Theorem 1 of [1]) tells us that D is pure $r-1$ dimensional. The proof of this theorem is based on Zariski's Jacobian Criterion and hence it is expected that this theorem could be generalized to the case when k is perfect, but if k is imperfect then one would have to replace the Jacobian Criterion by Zariski's Mixed Jacobian Criterion and that would *a priori* necessitate replacing the Jacobian determinant J in the proof of Theorem 1 of [1] by the various $m \times m$ subdeterminants of a certain $m \times M$ Jacobian matrix with $M > m$; hence one cannot predict whether the purity of the branch locus holds true for imperfect ground fields and this point should be looked into. However, it will be a consequence of Theorem 4 proved in the next section that if K^* is cyclic extension of prime degree q which is different from the characteristic of the arbitrary ground field k and if k contains a

primitive q-th root then D is pure $r-1$ dimensional (see Theorem 7 of the next section).

6. Kummer extensions of unique factorization domains. The following is the main theorem of this section.

THEOREM 4. *Let R be a unique factorization domain with quotient field K of characteristic p (which may or may not be zero). Let K^* be a cyclic extension of K of prime degree $q \neq p$, $(q > 1)$. Assume that K contains a primitive q-th root of unity. Let R^* be the integral closure of R in K^*. Let z be a primitive element of K^*/K for which $z^q = x \in R$ (such elements z do exist). Let $x = dy^q \prod_{i=1}^{s} x_i^{n_i}$ where $y \in R$; $x_1, x_2, \cdots, x_s$ are distinct (i.e. pairwise coprime) irreducible non-units in R, $0 < n_i < q$, and d is a unit in R. Let $jn_i = t_{ij}q + m_{ij}$ with integers t_{ij} and m_{ij} such that $t_{ij} \geqq 0$ and $0 < m_{ij} < q$ for $j = 1, 2, \cdots, q-1$. Let*

$$g_j = z^j y^{-j} \prod_{i=1}^{s} x_i^{-t_{ij}} \qquad \text{for } j = 1, 2, \cdots, q-1.$$

Then:

(I) $R^* = R + g_1R + g_2R + \cdots + g_{q-1}R$ *where the sum is R-direct; and*[2]

(II) $$D(1, g_1, g_2, \cdots, g_{q-1}) = (-1)^J q^q d^{q-1} (x_1 x_2 \cdots x_s)^{q-1},$$

where $J = (q-1)/2$ if $q > 2$ and $J = 0$ if $q = 2$; and hence

$$D(R^*/R) = (\prod_{i=1}^{s} x_i^{q-1})R.$$

Proof. Primitive elements z of K^*/K for which $z^q = x \in K$ exist since K contains primitive q-th roots of unity [$z^* \in K^*$ is another primitive element of K^*/K for which $z^{*q} \in K$ if and only if $z^* = uz^i$ with $0 \neq u \in K$ and $0 < i < q$]. Since K is the quotient field of R we may arrange matters so that $x \in R$.

Obviously $K^* = K + g_1K + g_2K + \cdots + g_{q-1}K$, (direct K-sum). Now

$$g_j^q = z^{jq} y^{-jq} \prod_{i=1}^{s} x_i^{-t_{ij}q} = z^{jq} y^{-jq} (\prod_{i=1}^{s} x_i^{-t_{ij}q - m_{ij}}) (\prod_{i=1}^{s} x_i^{m_{ij}})$$

$$= z^{jq} y^{-jq} (\prod_{i=1}^{s} x_i^{-jn_j}) (\prod_{i=1}^{s} x_i^{m_{ij}}) = (z^q / dy^q \prod_{i=1}^{s} x_i^{n_i})^j d^j \prod_{i=1}^{s} x_i^{m_{ij}} = d^j \prod_{i=1}^{s} x_i^{m_{ij}} \in R.$$

[2] We are using the following notation: $D(R^*/R)$ = the discriminant ideal of R^* over R = the ideal generated in R by all the K-discriminants $D(u_1, u_2, \cdots, u_q)$ of the various K-bases $(u_1, u_2, \cdots, u_q)$ of K^* consisting of elements in R^*.

Therefore $g_j \in R^*$ for $j=1,2,\cdots,q-1$, and hence $(R+g_1R+g_2R+\cdots+g_{q-1}R) \subset R^*$ where the sum is R-direct. Thus to complete the proof of (I) we have to show that $R^* \subset (R+g_1R+g_2R+\cdots+g_{q-1}R)$.

Let $h=z/y$. Then h is a primitive element of K^*/K with minimal monic polynomial X^q-f where $f=d\prod_{i=1}^{s} x_i^{n_i} \in R$. Since q is prime and since $0<n_i<q$, there exist integers M_i, N_i such that $0<M_i<q$ and $M_in_i+N_iq=1$. Let $h_i=h^{M_i}x_i^{N_i}$. Then h_i is a primitive element of K^*/K with minimal monic polynomial X^q-f_i with $f_i=x_iF_i$ where F_i is an element of R not divisible by x_i. Let v_i be the real discrete valuation of K which is given by the condition that for r in R we have $v_i(r)=$ the exponent of the highest power of x_i which divides r in R. Then $v_i(h_i^q)=1$ and hence v_i has a unique extension w_i to K^*. We normalize w_i so that $w_i(h_i)=1$. Then $w_i(k)=qv_i(k)$ for any k in K.

Now discriminant $(X^q-f)=(-1)^Jq^qf^{q-1}$. Therefore (see pages 79-80 of [7]), given $0 \neq a^* \in R^*$ we can find elements a_i in K such that

$$(1) \qquad a^*=a_0+a_1h+a_2h^2+\cdots+a_{q-1}h^{q-1} \text{ and } f^{q-1}a_j \in R.$$

Suppose, if possible, that $w_i(a_jh^j)=w_i(a_kh^k)$ for some $j \neq k$ with $a_j \neq 0 \neq a_k$. Then $n_i(j-k)=w_i(h^{j-k})=w_i(a_k/a_j)=qv_i(a_k/a_j) \equiv 0 \pmod q$ which is a contradiction since n_i and $(j-k)$ are both prime to q. Therefore no two non-zero terms on the right hand side of (1) are of equal w_i value. Since $a^* \in R^*$, $w_i(a^*) \geqq 0$ and hence $w_i(a_jh^j) \geqq 0$ for $j=1,2,\cdots,q-1$, (if $a_j=0$ then $w_i(a_jh^j)=\infty$). Let $b_j=a_j\prod_{i=1}^{s} x_i^{t_{ij}}$. Then

$$(2) \qquad a^*=b_0+b_1g_1+b_2g_2+\cdots+b_{q-1}g_{q-1} \text{ and } f^{q-1}b_j \in R \text{ for } j=1,2,\cdots,q-1.$$

Now $w_i(h^j)=(1/q)w_i(h^{qj})=v_i(h^{qj})=v_i(f^j)=t_{ij}q+m_{ij}$. Therefore

$$qv_i(a_j)=w_i(a_j) \geqq -t_{ij}q-m_{ij}, \text{ i.e., } v_i(a_j) \geqq -t_{ij}-(m_{ij}/q).$$

Since $0<(m_{ij}/q)<1$ we must have $v_i(a_j) \geqq -t_{ij}$. Hence $v_i(b_j)=v_i(a_j)+t_{ij} \geqq 0$, i.e., x_i does not occur in the denominator of b_j. This is true for $i=1,2,\cdots,s$. Since $(dx_1^{n_1}x_2^{n_2}\cdots x_s^{n_s})^{q-1}b_j \in R$, we must have that $b_j \in R$ for $j=0,1,\cdots,q-1$. Hence in view of the equation (2) we have that $R^* \subset (R+g_1R+g_2R+\cdots+g_{q-1}R)$ and hence that $R^*=(R+g_1R+g_2R+\cdots+g_{q-1}R)$.

Since $z^j=y^j(\prod_{i=1}^{s} x_i^{t_{ij}})g_j$ and since X^q-x is the minimal monic polynomial of z over K we have that

$$(-1)^J q^q x^{q-1} = \text{Discriminant of } (X^q - x) = D(1, z, z^2, \cdots, z^{q-1})$$

$$= (\prod_{j=1}^{q-1} [y^j \prod_{i=1}^{s} x_i^{t_{ij}}])^2 D(1, g_1, g_2, \cdots, g_{q-1})$$

$$= y^{q(q-1)} (\prod_{j=1}^{q-1} \prod_{i=1}^{s} x_i^{t_{ij}})^2 D(1, g_1, g_2, \cdots, g_{q-1})$$

Therefore $D(1, g_1, g_2, \cdots, g_{q-1}) = (-1)^J q^q d^{q-1} \prod_{i=1}^{s} x_i^{e_i}$, where

$$e_i = (q-1)n_i - 2\sum_{j=1}^{q-1} t_{ij} \geqq 0.$$

Let d_u be a unit in R; y_u an element in R; and $x_{u1}, x_{u2}, \cdots, x_{us_u}$ with $x_{u1} = x_u$ be distinct (i.e. pairwise coprime) irreducible non-units in R. Let $h_u = d_u y_u^q \prod_{i=1}^{s_u} x_{ui}^{n_{ui}}$ where n_{ui} are integers such that $0 < n_{ui} < q$. Then $n_{u1} = 1$. Let t_{uij} and m_{uij} be integers with $t_{uij} \geqq 0$ and $0 < m_{uij} < q$ such that $jn_{ui} = t_{uij} + m_{uij}$ for $j = 1, 2, \cdots, q-1$. Let $g_{uj} = h_u{}^j y_u{}^{-j} \prod_{i=1}^{s_u} x_{ui}{}^{-t_{uij}}$. Replacing z by h_u we coiclude that $R^* = (R + g_{u1}R + g_{u2}R + \cdots + g_{u(q-1)}R)$ and

$$D(1, g_{u1}, g_{u2}, \cdots, g_{u(q-1)}) = (-1)^J q^q d_u{}^{q-1} \prod_{i=1}^{s_u} x_{ui}^{e_{ui}}$$

where $e_{ui} = (q-1)n_{ui} - 2\sum_{j=1}^{q-1} t_{uij} \geqq 0$. Since $n_{u1} = 1$ we must have $t_{u1j} = 0$ for $j = 1, 2, \cdots, q-1$. Therefore $e_{u1} = (q-1)n_{u1} = q-1$. Since $(1, g_1, g_2, \cdots, g_{q-1})$ and $(1, g_{u1}, g_{u2}, \cdots, g_{u(q-1)})$ are both R-bases of R^* we must have $D(1, g_{u1}, g_{u2}, \cdots, g_{u(q-1)}) = d_u{}^* D(1, g_1, g_2, \cdots, g_{q-1})$ where $d_u{}^*$ is a unit in R. Therefore for $u = 1, 2, \cdots, s$, we have that

$$e_u = v_u(D(1, g_1, g_2, \cdots, g_{q-1})) = v_u(D(1, g_{u1}, g_{u2}, \cdots, g_{u(q-1)}) = e_{u1} = q-1.$$

Q. E. D.

COROLLARY. *The above proof yields the arithmetical result:*

$$(q-1)n_i - 2\sum_{j=1}^{q-1} t_{ij} = q-1.$$

A direct proof of this corollary: Since q is prime and since $0 < n_i < q$, it follows that $(m_{i1}, m_{i2}, \cdots, m_{i(q-1)})$ must be a permutation of $(1, 2, \cdots, q-1)$. Hence if we let all the summations Σ be taken over $j = 1, , \cdots, q-1$, we obtain:

$$\Sigma j n_i = \Sigma t_{ij} q + \Sigma m_{ij} = q\Sigma t_{ij} + q(q-1)/2.$$

Therefore

$$q\Sigma t_{ij} = -[q(q-1)/2] + (\Sigma j)n_i = [q(q-1)/2](-1+n_i).$$

Therefore

$$2\Sigma t_{ij} = (q-1)(n_i-1) = (q-1)n_i - (q-1).$$

Therefore

$$(q-1)n_i - 2\Sigma t_{ij} = q-1.$$

THEOREM 5. *In the notation of Theorem* 4 *assume that* R *is a local domain with maximal ideal* M *and that* $s \geqq 1$. *Then* R^* *is also a local domain. Let* B *be a basis of* M. *Then* $(B, g_1, g_2, \cdots, g_{q-1})R^* = M^*$ *where by* M^* *we are denoting the maximal ideal in* R^*.

Proof. We shall use the notation of the proof of Theorem 4. Since v_i has a unique extension to K^*, there exists a unique local domain in K^* lying above R and it must be R^*. Now for $i = 1, 2, \cdots, q-1$ we have that $g_i \in M$ and hence $g_i \in M^*$. Therefore $(B, g_1, g_2, \cdots, g_{q-1})R^* \subset M^*$. Let $m^* \in M^*$. Then $m^* \in R^*$ and hence $m^* = m_0 + m_1 g_1 + m_2 g_2 + \cdots + m_{q-1}g_{q-1}$ with $m_i \in R$. Since $g_i \in M^*$ we have

$$m_0 = m^* - m_1 g_1 - m_2 g_2 - \cdots - m_{q-1}g_{q-1} \in M^* \cap R = BR$$

and hence $m^* \in (B, g_1, g_2, \cdots, g_{q-1})R^*$.

THEOREM 6. *In the notation of Theorem* 5 *assume that* R *is noetherian and regular and that* $s \geqq 2$. *Then* R^* *is non-regular.*

Proof. We shall use the notation of the proof of Theorem 4. Let $A_0, A_1, \cdots, A_H$ be a minimal basis of M. Suppose, if possible, that there exist elements $a_1, a_2, \cdots, a_H$, $b_1, b_2, \cdots, b_{q-1}$ in R^* such that

$$A_0 = \sum_{i=1}^{H} a_i A_i + \sum_{i=1}^{q-1} b_i g_i. \tag{1}$$

Let a_{ij}, b_{ij} be elements in R such that

$$a_i = a_{i0} + \sum_{j=1}^{q-1} a_{ij} g_j \quad \text{and} \quad b_i = b_{i0} + \sum_{j=1}^{q-1} b_{ij} g_j. \tag{2}$$

Now $h^i h^j = h^{i+j}$ if $i+j < q$ and $h^i h^j = h^{i+j-q} f$ if $i+j \geqq q$. Therefore

$$g_i g_j = g_{i+j} \prod_{k=1}^{s} x_k^{t_{k,i+j-t_{k,i}-t_{k,j}}} \text{ if } i+j < q$$

and

$$g_i g_j = g_{i+j-q}\left(\prod_{k=1}^{s} x_k^{t_{k,i+j-q-t_{k,i-t}\ ,j}}\right) f = d g_{i+j-q}\left(\prod_{k=1}^{s} x^{n_k+t_{k,i+j-q-t_{k,i}-t_{k,j}}}\right), \text{ if } i+j \geqq q.$$

Therefore $a_iA_i = a_{i0}A_i + \sum_{j=1}^{q-1} a_{ij}A_ig_j$ and

$$b_ig_i = \sum_{p=0}^{i-1} db_{i,q-i+p}\Big(\prod_{k=1}^{s} x_k^{n_k+t_{k,p}-t_{k,i}-t_{k,q-i+p}}\Big)g_p$$
$$+ \sum_{p=i}^{q-1} b_{i,p-i}\Big(\prod_{k=1}^{s} x_k^{t_{k,p}-t_{k,i}-t_{k,p-i}}\Big)g_p\text{; [We take } t_{k,0}=0 \text{ and } g_0=1\text{].}$$

Hence by (1) we get

$$A_0 = \sum_{i=1}^{H} a_{i0}A_i + d\sum_{i=1}^{q-1} b_{i,q-i}\Big(\prod_{k=1}^{s} x_k^{n_k-t_{k,i}-t_{k,q-i}}\Big) \tag{3}$$

Now $in_k = t_{k,i}q + m_{k,i}$ and $(q-i)n_k = t_{k,q-i}q + m_{k,q-i}$ imply that

$$qn_k = (t_{k,i} + t_{k,q-i})q + m_{k,i} + m_{k,q-i}$$

and hence $m_{k,i} + m_{k,q-i} = q$ and this in turn implies that $n_k - t_{k,i} - t_{k,q-i} = 1$. Therefore by (3) we get

$$A_0 = \sum_{i=1}^{H} a_{i0}A_i + d\Big(\prod_{k=1}^{s} x_k\Big)\sum_{i=1}^{q-1} b_{i,q-i}.$$

Comparing leading forms, we conclude that this is a contradiction since $s \geqq 2$.

Similarly for $i = 1, 2, \cdots, H$, there do not exist elements a_i, b_i in R for which $A_i = \sum_{i=0,1,\cdots,i-1,\,i+1,\cdots,H} a_iA_i + \sum_{i=1}^{q-1} b_ig_i$.

Now suppose, if possible, that there exist elements a_i, b_i in R^* such that $g_1 = h = \sum_{i=0}^{H} a_iA_i + \sum_{i=2}^{q-1} b_ig_i$. Using expressions (2) for a_i, b_i we get

$$1 = \sum_{i=0}^{H} a_{i1}A_i + d\sum_{i=2}^{q-1} b_{i,q-i+1}\Big(\prod_{k=1}^{s} x_k^{n_k-t_{k,i}-t_{k,q-i+1}}\Big). \tag{4}$$

Now $in_k = t_{k,i}q + m_{k,i}$ and $(q-i+1)n_k = t_{k,q-i+1}q + m_{k,q-i+1}$ and hence $qn_k + n_k = (t_{k,i} + t_{k,q-i+1})q + m_{k,i} + m_{k,q-i+1}$. Therefore

$$q(n_k - t_{k,i} + t_{k,q-i+1}) = -n_k + m_{k,i} + m_{k,q-i+1}. \tag{5}$$

Replacing, if necessary, z by h_1, we may assume that $n_1 = 1$. Then $m_{1,i} = i$ and $m_{1,q-i+1} = q - i + 1$ so that $-n_1 + m_{1,i} + m_{1,q-i+1} = -1 + i + q - i + 1 = q$. Hence by (5), $n_1 - t_{1,i} - t_{1,q-i+1} = 1$. Hence by (4) we conclude that $1 \in M^*$ which is a contradiction.

Therefore the elements $A_0, A_1, \cdots, A_H$, $h = g_1$, of M^* are linearly independent modulo M^{*2} and hence

$$\dim(M^*/M^{*2}) > H + 1 = \dim R = \dim R^*.$$

Therefore R^* is non-regular.

THEOREM 7. *Let V be a projective normal model of an r-dimensional algebraic function field K/k of characteristic p. Let K^* be a cyclic extension of K of prime degree $q \neq p$, $(q > 1)$ and let V^* be a K^*-derived normal model of V and assume that k contains a primitive q-th root of unity. Let D be the branch locus on V of the rational transformation from V^* onto V. Let P be a simple point of V. Then the component of D passing through P is pure $(r-1)$-dimensional (or empty). If V is non-singular then D is pure $(r-1)$ dimensional (or empty).*

Proof. Let R be the quotient ring of P on V. Then R is a unique factorization domain. Let R^* be the integral closure of R in K^*. Then the discriminant ideal $D(R^*/R)$ of R^* over R defines the component of D passing through P. By Theorem 4, $D(R^*/R)$ is either the unit ideal or is a principal ideal other than the unit ideal. Hence the result.

7. Simultaneous resolution for quadratic and cubic cyclic extensions. The purpose of this section is to prove

THEOREM 8. *Let K be a two dimensional algebraic function field over a perfect ground field k of characteristic p and let K^* be a finite algebraic extension of K. Assume that either K^*/K is quadratic and $p \neq 2$ or that K^*/K is cubic cyclic, $p \neq 3$ and k contains a primitive cube root of unity. Then there exists a non-singular projective model V of K/k whose K^*-normalization is also non-singular and the branch locus on V is a non-singular curve (or is empty).*

Since k is perfect, K/k has non-singular models [4]; we remark that this the only thing for which the assumption of perfectness of k will be used and hence that the theorem remains true if the assumption of perfectness of k is replaced by the assumption that K/k has non-singular models.

Now let V be a non-singular projective model of K/k, and let V^* be a K^*-normalization of V. By Theorem 6, the branch locus on V is a curve D. Let $D_1, D_2, \cdots, D_n$ be the irreducible components of D. Let P be a singular point of D_1 and let V' be the surface obtained from V by applying a local quadratic transformation centered at P. Let D_i' be the curve on V' which corresponds to D_i and let L be the (non-singular fundamental) curve on V' which is the total image of P. Let V'^* be a K^*-normalization of V'. Then the branch locus on V' (for the transformation between V' and V'^*) consists either of $D_1', D_2', \cdots, D_n'$ or of $D_1', D_2', \cdots, D_n', L$. In an obvious sense L intersects any of the curves D_i' normally.

Let $P_1, P_2, \cdots, P_N$ be the set of singularities of $D_1, D_2, \cdots, D_n$. Let V_1 be the surface obtained from V by applying a quadratic transformation centered at $(P_1, P_2, \cdots, P_N)$.[3] Let V_2 be the surface obtained from V_1 as V_1 is obtained from V. Let $V_1, V_2, V_3, \cdots$ be the sequence of surfaces defined in this manner. In view of Theorem 4 of Part II of Zariski [11], it follows from the above remarks that for some t, all the irreducible components of the branch curve on V_t (for the transformation between V_t and its K^*-normalization) are non-singular. We may assume that this is so already for V, i.e. that the curves D_i are all non-singular.

Now let $Q_1, Q_2, \cdots, Q_M$ be the points of V which are common to more than one component of D, (i.e., the singular points of D). Let $V^{(1)}$ be the surface obtained from V by applying a quadratic transformation centered at $(Q_1, Q_2, \cdots, Q_M)$. Let $V^{(2)}$ be the surface obtained from $V^{(2)}$ as $V^{(1)}$ is obtained from V. Let $V^{(1)}, V^{(2)}, V^{(3)}, \cdots$ be the sequence of surfaces thus defined. Now suppose for instance that $Q_1 \in D_1$; let R be the quotient ring of Q_1 on V, let w be the rank two valuation of K/k having center at Q_1 composed with the real discrete valuation given by D_1 (w is unique since Q_1 is simple for D_1). Let R_i be the i-th quadratic transform of R along w. By Lemma 12 of [2], $\bigcup_{i=1}^{\infty} R_i = R_w$. In view of this, it is clear that we have proved the following proposition.

PROPOSITION 4. *In the notation of Theorem 8, let V be a non-singular projective model of K/k, let $P_1, P_2, \cdots, P_N$ be the singular points of the branch curve D on V (of the transformation between V and its K^*-normalization). Let V_1 be the surface obtained from V by applying a quadratic transformation centered at $(P_1, P_2, \cdots, P_N)$. Let V_2 be the surface obtained from V_1 as V_1 is obtained from V. Let $V_1, V_2, V_3, \cdots$ be the sequence of surfaces thus defined. Let D_i be the branch curve on V_i for the transformation between V_i and its K^*-normalization. Then there exists an integer n such that for any $i \geqq n$, the only singularities of D_i are two-fold normal crossings (Definition 4 of [1]).*

PROPOSITION 5. *Let (R, M) be an r-dimensional regular noetherian local domain, with quotient field K and let $(x_1, x_2, \cdots, x_r)$ be a basis of M. Let $K^* = K(z)$ with $z^q = dx_1^n$ where q is a prime number, n is a positive integer not divisible by q and d is a unit in R. Then there is only one local ring R^* in K^* lying above R and R^* is regular.*

[3] See part (2) of the proof of Theorem 3 of Section 4.

Proof. The proof follows from the argument given in the proof of Theorem 2 of [3].

In view of Proposition 4 the proof of Theorem 8 in the quadratic and the cubic cases follows respectively from Theorems 9 and Theorems 10 below.

THEOREM 9. *Let V be a non-singular surface with quotient field K/k of characteristic $\neq 2$ and K^* a quadratic extension of K. Let V^* be a K^*-normalization of V and assume that all the singularities $P_1, P_2, \cdots, P_n$ of the branch curve D on V for the transformation between V and V^* are ordinary double points. Let V' be the surface obtained from V by applying a quadratic transformation centered at $(P_1, P_2, \cdots, P_n)$ and let D' be the branch curve on V' for the transformation between V' and its K^*-normalization V'^*. Then V', V'^* and D' are all non-singular.*

Proof. Let P be one of the singular points of D and let (R, M) be the quotient ring of P on V. By Theorem 4 there exists a primitive element z of K^*/K such that $z^2 = dxy$ where (x, y) is a basis of M, d is a unit in R and $xy = 0$ is the local equation of D at P. We may replace x by dx and assume that $z^2 = xy$. Let (R_1, M_1) be an immediate two dimensional quadratic transform (Definition 3 of[2]) of R and let v be a valuation of K/k having center M_1 in R_1. We shall now show that there exists a primitive element z_1 of K^*/K such that either $z_1^2 = d_1$ or $z_1^2 = x_1$ where d_1 is a unit in R_1 and (x_1, y_1) is a basis of M_1 and in view of Theorem 4 and Proposition 5, the proof of the present theorem will then follow from this. *Case* 1, $v(x) > v(y)$. Let $y_1 = y$ and $x_1 = x/y$. Let $z = z/y$. Then (x_1, y_1) is a basis of M_1, z_1 is a primitive element of K^*/K and $z_1^2 = x_1$. *Case* 2, $v(x) < v(y)$. Because of symmetry in x and y the proof in this case follows from Case 1. *Case* 3, $v(x) = v(y)$. Let $d_1 = x/y$ and $z_1 = z/y$. Then d_1 is a unit in R_1, z_1 is a primitive element of K^*/K and $z_1^2 = d_1$.

THEOREM 10. *Let the notation be as in Theorem 9, except assume that K^*/K is cubic cyclic, k is of characteristic $\neq 3$ and contains a primitive cube root of unity. Let V'' be the surface obtained from V' as V' was obtained from V. Let D'' be the branch curve on V'' for the transformation between V'' and its K^*-normalization V''^*. Then V'', V''^* and D'' are all non-singular.*

Proof. Let P'' be an arbitrary point of D'', let P' be the point of V' which corresponds to P'' and let P be the point of V which corresponds to P'. Then $P' \in D'$ and $P \in D$. Let (R_2, M_2) and (R_1, M_1) be the quotient rings of P'' and P' on V' and V'' respective and let v be a valuation of K/k

having center M_2 in R_2. In view of Theorem 4 and Proposition 5, it is enough to show that P'' is a simple point of D''.

If P' is a simple point of D' then $R_2 = R_1$ and P'' is a simple point of D''. Now assume that P' is singular for D'. Then P' and P are ordinary double points of D' and D respectively. By Theorem 4, we can find a primitive element z of K^*/K such that either (Case 1) $z^3 = dxy^2$ or (Case 2) $z^3 = dx^2y$ or (Case 3) $z^3 = dxy$ where d is a unit in R and (x, y) is a basis of M.

Suppose, if possible, that we have Case 1, i.e., $z^3 = dxy^2$. We have the following three subcases. *Subcase* 1.1: $v(x) < v(y)$. Let $z_1 = z/x$, $x_1 = x$, and $y_1 = y/x$. Then z_1 is a primitive element of K^*/K, (x_1, y_1) is a basis of M_1 and ${z_1}^3 = dy_1^2$ and hence by Theorem 4, P' is a simple point of D' which is a contradiction. *Subcase* 1.2: $v(x) > v(y)$. Let $z_1 = z/y$, $x_1 = x/y$ and $y_1 = y$. Then z_1 is a primitive element of K^*/K, (x_1, y_1) is a basis of M_1 and ${z_1}^3 = dx_1$ and hence by Theorem 4, P' is a simple point of D' which is a contradiction. *Subcase* 1.3: $v(x) = x(y)$. Then $z_1 = z/y$ is a primitive element of K^*/K and ${z_1}^3 = e$ where $e = dx/y$ is a unit in R_1; hence P' is not a point of D' which is a contradiction.

Thus Case 1 is not possible and similarly Case 2 is not possible either. Thus we are left with Case 3, i.e., we have $z = dxy$. Suppose, if possible, that $v(x) = v(y)$. Let $x_1 = x$ and $d_1 = dy/x$. Then x_1 is part of a basis (x_1, y_1) of M_1, d_1 is a unit in R_1 and $z^3 = d_1{x_1}^2$; hence by Theorem 4, P' is a simple point of D' which is a contradiction. Therefore $v(x) \neq v(y)$. Because of symmetry in x and y, it is enough to complete the argument in case $v(x) > v(y)$. Let $y_1 = y$ and $x_1 = x/y$. Then (x_1, y_1) is a basis of M_1 and $z^3 = dx_1{y_1}^2$. The argument which implied the impossibility of Case 1 now tells us that $z^3 = dx_1{y_1}^2$ implies that P'' is a simple point of D''.

8. Simultaneous nonresolvability for cyclic extensions of minimal models.

DEFINITION. *Let K/k be an algebraic function field and let K^* be a finite algebraic extension of K. If there does not exist any non-singular projective model of K/k with a non-singular K^*-normalization then we shall say that (K, K^*) is simultaneously non-resolvable.*

Now let (R, M) be a regular two dimensional (noetherian) local domain with quotient field K. Let (x_0, y_0) be a basis of M. Let a be a fixed positive integer and let $b = a + 1$. Let

$$
\begin{array}{ll}
x_1 = x_0/y_0, & y_1 = y_0; \\
x_2 = x_1/y_1, & y_2 = y_1; \\
\cdots\cdots & \\
x_{b-1} = x_{b-2}/y_{b-2}, & y_{b-1} = y_{b-2}; \\
x_b = x_{b-1}, & y_b = y_{b-1}/x_{b-1}; \\
x_{b+1} = x_b/y_b, & y_{b+1} = y_b; \\
x_{b+2} = x_{b+1}/y_{b+1}, & y_{b+2} = y_{b+1}; \\
\cdots\cdots & \\
x_{2b-1} = x_{2b-2}/y_{2b-2}, & y_{2b-1} = y_{2b-2}; \\
x_{2b} = x_{2b-1}, & y_{2b} = y_{2b-1}/x_{2b-1}. \\
\cdots\cdots &
\end{array} \tag{1}
$$

i.e. for $n = 0, 1, 2, \cdots$ we have set

$$
\begin{array}{l}
x_{nb+i} = x_{nb+i-1}/y_{nb+i-1}, \quad y_{nb+i} = y_{nb+i-1} \quad \text{for } i = 1, 2, \cdots, b-1 \\
\text{and } x_{nb+b} = x_{nb+b-1}, \quad y_{nb+b} = y_{nb+b-1}/x_{nb+b-1};
\end{array} \tag{2}
$$

in other words

$$
\begin{array}{l}
x_{nb+i} = x_{nb}/y_{nb}{}^i, \quad y_{nb+i} = y_{nb} \quad \text{for } i = 1, 2, \cdots, b-1, \\
\text{and } x_{(n+1)b} = x_{nb}/y_{nb}{}^{b-1}, \quad y_{(n+1)b} = y_{nb}{}^b/x_{nb}.
\end{array} \tag{3}
$$

Then there is a unique two dimensional h-th quadratic transform (Definition 3 of [2]) (R_h, M_h) of (R, M) such that $M_h = (x_h, y_h)R_h$. More explicitly we can define (R_h, M_h) either by induction as:

For $n = 0, 1, 2 \cdots$ let

$$S_{nb+i} = R_{nb+i-1}[x_{nb+i}], \quad N_{nb+i} = (x_{nb+i}, y_{nb+i})S_{nb+i} \text{ for } i = 1, 2, \cdots, b-1,$$

and

$$S_{nb+b} = R_{nb+b-1}[y_{nb+b}], \quad N_{nb+b} = (x_{nb+b}, y_{nb+b})S_{nb+b};$$

and then

$$R_{nb+i} = (S_{nb+i})_{N_{nb+i}}, \quad M_{nb+i} = N_{nb+i}R_{nb+i} \text{ for } i = 1, 2, \cdots, b.$$

Or directly as:

$$S_t{}^* = R[x_t, y_t], \quad N_t{}^* = (x_t, y_t)S_t{}^*, \quad R_t = (S_t{}^*)_{N_t{}^*} \text{ and } M_t = N_t{}^*R_t{}^*.$$

Anyway, R_{t+1} is an immediate quadratic transform of R_t for $t = 0, 1, 2, \cdots$ where $R_0 = R$.

By Lemma 12 of [2], $\bigcup_{i=1}^{\infty} R_i = R_v$ where v is a valuation of K having center M_i in R_i. It follows from the definition of the sequence (x_1, y_1),

$(x_2, y_2), \cdots$ that $v(x)$ and $v(y)$ are comparable [i.e., there exist positive integers A and B such that $Av(x) > v(y)$ and $Bv(y) > v(x)$] and that (the real number) $Q = v(x)/v(y)$ is an irrational number given by the periodic continued fraction expansion (of period two):

$$Q = a + 1/(1 + 1/(a + 1/(1 + 1/(a + \cdots.$$

Hence, in view of the results of [2], v is a real valuation of rational rank two and $R_v/M_v = R/M = R_i/M_i$ for any i. If we set $v(y) = 1$ and $v(x) = Q$, then the value group of v consists of the real numbers of the form $m + nQ$ where m and m are arbitrary integers.

THEOREM 11. *Let (R, M) be a two dimensional regular (noetherian) local domain with quotient field K of characteristic p and let (x, y) be a basis of M. Let q be a prime number such that $q > 3$ and $q \neq p$. Let $a = q - 4$, $b = a + 1 = q - 3$, $x_0 = x$ and $y_0 = y$. Let v be the valuation of K as constructed above. Let K^* be the root field over K of the polynomial $Z^q - xy^2$. Let (S, N) be any two dimensional regular (noetherian) local domain with quotient field K such that $R_v \supset S \supset R$, $M_v \cap S = N$ and $M = R \cap N$. Then there is only one local ring S^* in K^* lying above S and S^* is not regular.*

Proof. We shall use the notation introduced above in the construction of v. By Theorem 3 of [2] there exists an integer h such that $S = R_h$ and $N = M_h$. Thus we have to prove that for any integer t, there is only one local ring R_t^* in K^* lying above and R_t^* is non-regular.

Let H be a root of $Z^q - xy^2 = 0$. Then H is a primitive element of K^*/K and K^*/K is cyclic of degree q. Let $F = xy^2$. Let us rewrite equations (3) as:

For $n = 0, 1, 2, \cdots$ and $i = 1, 2, \cdots, b-1$,

$$(4) \qquad \begin{aligned} &x_{nb} = (x_{nb+i})(y_{nb+i})^i = (x_{nb+b})^b (y_{nb+b})^{b-1}, \qquad y_{nb} = y_{nb+i} = x_{nb+b} y_{nb+b}, \\ &x_{nb+b-1} = x_{nb+b} \qquad y_{nb+b-1} = x_{nb+b} y_{nb+b}. \end{aligned}$$

Let $F_i = F$ and $H_i = H$ for $i = 0, 1, \cdots, b$. Then for $i = 1, 2, \cdots, b-1$ we have $F_i = xy^2 = (x_i y_i^i)(y_i)^2 = x_i y_i^{i+2}$ with $0 < i + 2 \leqq (b-1) + 2 = b + 1 < q$, and $F_b = (x_b y_b^{b-1})(x_b y_b)^2 = x_b^{b+2} y_b^{b+1}$ with $0 < b + 2 < q$ and $0 < b + 1 < q$.

For $i = 1, 2, \cdots, b-1$ let $F_{b+i} = F_b (y_{b+i})^{-iq}$ and $H_{b+i} = H_b (y_{b+i})^{-i}$. Then H_{b+i} is a primitive element of K^*/K and

$$(H_{b+i})^q = F_{b+i} = F_b(y_{b+i})^{-iq} = x_b{}^{b+2}y_b{}^{b+1}(y_{b+i})^{-iq}$$
$$= (x_{b+i}y_{b+i}{}^i)^{b+2}(y_{b+i})^{b+1}(y_{b+i})^{-iq}$$
$$= (x_{b+i})^{b+2}(y_{b+i})^{i(b+2)+b+1-i(b+3)} = (x_{b+i})^{b+2}(y_{b+i})^{b+1-i}$$

and $0 < b+2 < q$, $0 < b+1-i < q$.

In particular, substituting $i = b-1$ we get

$$(H_{2b-1})^q = F_{2b-1} = (x_{2b-1})^{b+2}(y_{2b-1})^2.$$

Let $F_{2b} = F_{2b-1}(x_{2b})^{-q}$ and $H_{2b} = H_{2b-1}(x_{2b})^{-1}$. Then H_{2b} is a primitive element of K^*/K and

$$(H_{2b})^q = F_{2b} = F_{2b-1}(x_{2b})^{-q} = (x_{2b-1})^{b+2}(y_{2b-1})^2(x_{2b})^{-q}$$
$$= (x_{2b})^{b+2}(x_{2b}y_{2b})^2(x_{2b})^{-q} = (x_{2b})^{b+4-q}(y_{2b})^2 = x_{2b}(y_{2b})^2$$

i.e.,

$$(H_{2b})^q = F_{2b} = x_{2b}(y_{2b})^2. \tag{5}$$

Let $F_{2b+i} = F_{2b}$, $H_{2b+i} = H_{2b}$ for $i = 1, 2, \cdots, b$; $F_{3b+i} = F_{3b}(y_{3b+i})^{-iq}$, $H_{3b+i} = H_{3b}(y_{3b+i})^{-i}$ for $i = 1, 2, \cdots, b-1$; and $F_{4b} = F_{4b-1}(x_{4b})^{-q}$, $H_{4b} = H_{4b-1}(x_{4b})^{-1}$. Then in view of (5), the above considerations at once tell us that H_{2b+i} is a primitive element of K^*/K for $i = 1, 2, \cdots, 2b$; and

$$(H_{2b+i})^q = F_{2b+i} = x_{2b+i}(y_{2b+i})^{i+2} \text{ for } i = 1, 2, \cdots, b-1;$$
$$(H_{3b+i})^q = F_{3b+i} = (x_{3b+i})^{b+2}(y_{3b+i})^{b+1-i} \text{ for } i = 0, 1, \cdots, b-1;$$
$$(H_{4b})^q = F_{4b} = x_{4b}(y_{4b})^2.$$

And so on $\cdots$, i.e., by induction we define for $n = 0, 1, 2, \cdots$

$$F_{2nb+i} = F_{2nb}, \quad H_{2nb+i} = H_{2nb} \text{ for } i = 1, 2, \cdots, b;$$
$$F_{2nb+b+i} = F_{2nb+b}(y_{2nb+b+i})^{-iq}, \quad H_{2nb+b+i} = H_{2nb+b}(y_{2nb+b+i})^{-i}$$
$$\text{for } i = 1, 2, \cdots, b-1;$$

and

$F_{2nb+2b} = F_{2nb+2b-1}(x_{2nb+2b})^{-q}$, $H_{2bn+2b} = H_{2nb+2b-1}(x_{2nb+2b})^{-1}$; and we conclude that for any non-negative t, H_t is a primitive element of K^*/K and

$H_t{}^q = F_t = x_t{}^{A_t}y_t{}^{B_t}$ with $0 < A_t < q$ and $0 < B_t < q$. [Observe that $A_{2nb+i} = A_i$, $B_{2nb+i} = B_i$ for $n = 0, 1, 2, \cdots$ and $i = 1, 2, \cdots, 2b$]. Hence by Theorems 4 and 6 it follows that there is only one local ring $R_t{}^*$ in lying above R_t and $R_t{}^*$ is not regular. Q. E. D.

Now let K/k be a two dimensional algebraic function field and recall that a non-singular projective model V of K/k is called a *minimal model* of K/k if W is a non-singular projective model of K/k implies that the

(uniquely determined) birational transformation from W onto V is everywhere regular on W. In [14] Zariski has proved (at least when k is algebracically closed) that K/k has a minimal model if and only if there does not exist any intermediate field between k and K of which K is a simple transcendental extension.

THEOREM 12. *Let K/k be a two dimensional algebraic function field of characteristic p and assume that K/k has a minimal model. Let q be a prime number such that $q>3$ and $q\neq p$. Assume that k contains a primitive q-th root of unity. Then there exist q-cyclic extensions (lots of them) K^* of K such that (K,K^*) is simultaneously non-resolvable.*

Proof. Let V be the minimal model of K/k. Let (R,M) be the quotient ring of a point of V. Let (x,y) be a basis of M. Let K^* be a root field over K of the polynomial Z^q-xy^2. It then follows by Theorem 11 that (K,K^*) is simultaneously non-resolvable.

9. Partial generalizations to higher varieties. In this section we plan to give partial generalizations to higher varieties of the theorems proved in Sections 2, 3 and 7 for surfaces. We start off by deducing the following consequence (which is the weak local uniformization theorem for real valuations for an algebraically closed ground field of characteristic zero) of a result proved by Zariski in his uniformization paper [9].

THEOREM 13. *Let P be a simple point on an r-dimensional algebraic variety V with function field K/k where the ground field k is algebraically closed of characteristic zero. Let (R,M) be the quotient ring of P on V. Let v be a zero dimensional real valuation of K/k having center P on V and let u be a non-zero element of R. Then there exists an anti-regular transform V^* of V obtained from V by a sequence of monoidal transformations such that there is a minimal basis $(x_1^*,x_2^*,\cdots,x_r^*)$ of the maximal ideal M^* of the quotient ring R^* of the center P^* of v on V^* for which $u=x_1^{*a_1}x_2^{*a_2}\cdots x_h^{*a_h}d$ where $a_1,a_2,\cdots,a_h$ are non-negative integers, d is a unit in R^* and $v(x_1),v(x_2),\cdots,v(x_h)$ are rationally independent.*

It is clear that this theorem will follow from the following slightly more general proposition.

PROPOSITION 6. *Let (R,M) be an r-dimensional $(r>1)$ regular (noetherian) local domain with quotient field K. Assume that R contains a subfield k which is mapped isomorphically onto the residue field R/M by the canonical homomorphism and assume that k is algebraically closed of*

characteristic zero. Let $(x_1, x_2, \cdots, x_r)$ *be a basis of* M. *Let* v *be a real valuation of* K/k *having center* M *in* R *and of* R*-dimension zero,* (i.e., $k = R_v/M_v$). *Let* u *be a non-zero element in* M. *Then there exists a regular* r*-dimensional local domain* (R^*, M^*) *obtained from* R *by a sequence of monoidal transformations with respect to the basis* $(x_1, x_2, \cdots, x_r)$ *and the field* k (*see definition below*) *and a basis* $(x_1^*, x_2^*, \cdots, x_r^*)$ *of* M^* *such that* $u = dx_1^{*a_1} x_2^{*a_2} \cdots x_h^{*a_h}$ *where* d *is a unit in* R^*; $a_1, a_2, \cdots, a_h$ *are positive integers and* $v(x_1), v(x_2), \cdots, v(x_h)$ *are rationally independent.*

Proof. For $i = 1, 2, \cdots, r$ let $f_i(X_1, X_2, \cdots, X_r)$ be elements of $k[X_1, X_2, \cdots, X_r]$ of leading degree one such that if $g_{i1}X_1 + g_{i2}X_2 + \cdots + g_{ir}X_r$ with g_{ij} in k is the leading form of f_i then the r by r determinant $(g_{ij}) \neq 0$. Let $y_i = f_i(x_1, x_2, \cdots, x_r)$. Then $(y_1, y_2, \cdots, y_r)$ is a basis of M. Assume that $v(y_1) < v(y_i)$ for $i = 2, 3, \cdots, t$ with $t \leqq r$. Let $z_1 = y_1$, $z_i = y_i/y_1$ for $i = 2, 3, \cdots, t$ and $z_i = y_i$ for $i = t+1, t+2, \cdots, r$. Let $S = R[z_1, z_2, \cdots, z_r]$, $N = S \cap M_v$ and $R_1 = S_N$ and $M_1 = NR_1$. Then (R_1, M_1) is a regular r-dimensional local domain, $(z_1, z_2, \cdots, z_r)$ is a basis of M_1, $R_v \supset R_1 \supset R$, $M_v \cap R_1 = M_1$ and $M_1 \cap R = M$. We shall say that (R_1, M_1) is a first (or an immediate) monoidal transform of R along v with respect to the basis $(x_1, x_2, \cdots, x_r)$ and the field k. Let (R_2, M_2) be an immediate monoidal transform of R_1 along v with respect to the basis $(z_1, z_2, \cdots, z_r)$ and the field k. Then we shall say that R_2 is a second monoidal transform of R along v with respect to the basis $(x_1, x_2, \cdots, x_r)$ and the field k and so on.

Now let $A = k[x_1, x_2, \cdots, x_r]$, $Q = (x_1, x_2, \cdots, x_r)A$, $R' = A_Q$, $M' = QR'$, and let v' be the restriction of v to $k(x_1, x_2, \cdots, x_r)$. It is a direct consequence of what Zariski has proved in parts B, CI and CII of [9] that the present proposition holds for (R', M'); this special case of the proposition will be referred to as Z.

Let $E = \min(v(x_1), v(x_2), \cdots, v(x_r))$ and let $v(u) = D$. Fix a positive integer t such that $tE > D$. We can write $u = u_1 + u'$ where u_1 is a polynomial in $k[x_1, x_2, \cdots, x_r]$ of degree less than t and $u' \in M^t$. Then $v(u_1) = v(u) = D$ and $u = u_1 + b_2u_2 + \cdots + b_Tu_T$ where $b_i \in R$ and $u_2, u_3, \cdots, u_T$ are monomials of degree t in $x_1, x_2, \cdots, x_r$. Applying Z to the product $u_1u_2 \cdots u_T$ we conclude that there exists an n-th monoidal transform (R'^*, M'^*) of R' along v' with respect to the basis $(x_1, x_2, \cdots, x_r)$ and the field k such that $u_i = d_i x_1^{*a_{i1}} x_2^{*a_{i2}} \cdots x_H^{*a_{iH}}$ for $i = 1, 2, \cdots, T$ where the a_{ij} are non-negative integers, d_i is a unit in R'^*, $(x_1^*, x_2^*, \cdots, x_r^*)$ is a basis of M'^* and $v'(x_1^*), v'(x_2^*), \cdots, v'(x_H^*)$ are rationally independent.

Since $v'(u_i) > v'(u_1)$ for $i = 2, 3, \cdots, T$; invoking Theorem 2 of [9] we may arrange matters so that $a_{1j} < a_{ij}$ for $j = 1, 2, \cdots, H$ and $i = 2, 3, \cdots, T$. Let $S = R[x_1^*, x_2^*, \cdots, x_r^*]$, $N = (x_1^*, x_2^*, \cdots, x_r^*)$, $R^* = S_N$ and $M^* = NR^*$. Then evidently (R^*, M^*) is an n-th monoidal transform of R along v with respect to the basis $(x_1, x_2, \cdots, x_r)$ and the field k and $(x_1^*, x_2^*, \cdots, x_r^*)$ is a basis of M^*. Also it is clear that

$$u = dx_1^{*a_{11}} x_2^{*a_{12}} \cdots x_H^{*a_{1H}}$$

where d is a unit in R^*. Rearrange the $x_1^*, x_2^*, \cdots, x_H^*$ so that $a_{1i} \neq 0$ for $i = 1, 2, \cdots, h$ and $a_{1i} = 0$ for $i > h$. Setting $a_{1i} = a_i$ for $i = 1, 2, \cdots, h$ we obtain the required result.

PROPOSITION 7. *Let the notation be as in Proposition 6 and let w be an element of R_v. Then there exists R^* as described in Proposition 6 such that $w \in R^*$.*

Proof. Let $w = w_1/w_2$ with $w_1, w_2 \in R^*$. Applying Proposition 6 to the product $w_1 w_2$ and invoking Theorem 2 of Zariski [9] we obtain the result.

Now we give a partial generalization of Theorem 1 (Section 2) to higher varieties.

THEOREM 1A. *Let K be an n-dimensional algebraic function field over an algebraically closed ground field k of characteristic zero. Let K^* be a finite algebraic extension of K and let v^* be a zero dimensional real valuation of K^*/k. Let r be the rational rank of v^*. Then there exists a regular local ring (R^*, M^*) which is the quotient ring of the center of v^* on a projective model V^* of K^*/k such that if we set $S = R^* \cap K$, $Q = (M^* \cap K)$ and $Q^* = QR^*$ then we have the following:* (1) *S is normal domain with quotient field K and Q is the unique maximal ideal in S;* (2) *if $r = 1$ then rank $Q^* \geqq 2$;* (3) *if $r > 1$ then rank $Q^* \geqq r$;* (4) *if $r = n$ then V^* is a K^*-normalization of a projective normal model of K/k.*

Proof. Since v can be uniformized [9], there exists a regular local ring (R^*, M^*) which is the quotient ring of the center of v^* on a projective normal model of K^*/k. To arrange matters so that the transcendence degree of S over k is n we may either use the argument used in Theorem 1 together with Proposition 7 or we may invoke the fact that Zariski has proved the local uniformization theorem for v in projective form [9]. The rest of (1) is obvious. In view of Theorem 13, (2) follows by the argument used in Case 1 in the proof of Theorem 1.

Now assume that $r > 1$. Then r is also the rational rank of the K-

restriction of V^*. Fix elements $z_1, z_2, \cdots, z_r$ in K such that $v^*(z_1), v^*(z_2), \cdots, v^*(z_r)$ are rationally independent and positive. Either by Proposition 7 or by the projective form of the uniformization theorem [9], we may assume that the elements z_i are in R^*. Applying Theorem 13 to the product $z_1 z_2 \cdots z_r$ we may assume that $z_i = d_i x_1^{a_{i1}} x_2^{a_{i2}} \cdots x_h^{a_{ih}}$ where $(x_1, x_2, \cdots, x_n)$ is a basis of M^*, d_i is a unit in R^*, a_{ij} is a non-negative integer, and $v^*(x_1), v^*(x_2), \cdots, v^*(x_h)$ are rationally independent. Since $v^*(z_1), v^*(z_2), \cdots, v^*(z_r)$ are rationally independent we must have $h = r$ and for the r by r determinants $D = (a_{ij})$ we have $D \neq 0$. Hence there exist integers n_{ij} and units e_j in R^* such that $e_j x_j^D = z_1^{n_{1j}} z_2^{n_{2j}} \cdots z_r^{n_{rj}}$ for $j = 1, 2, \cdots, r$. Therefore the elements $x_1, x_2, \cdots, x_r$ belong to the radical of Q^* and hence rank $Q^* \geqq r$. This proves (3) and (4) follows from (3) and Proposition 1 of [4].

Proposition 3 of Section 3 can be generalized as follows:

PROPOSITION 3A. *Let K be an r-dimensional algebraic function field of characteristic zero and K^* a galois extension of K. Let v^* be a zero dimensional real valuation of K^* and let v be the K-restriction of v^*. Then there exists a regular local ring R which is the quotient ring of the center of v on a projective normal model V of K such that if R^* denotes the quotient ring of the center of v^* on a K^*-normalization of V then the splitting field of R^* over R coincides with the splitting field of v^* over v.*

Proof. The argument used in the proof of Theorem 1 of [4] tells us that in order that what is required should happen, it is sufficient that a certain finite number of elements of R_v be contained in R. Now this can be arranged either by using Proposition 7 or simply by invoking the fact that Zariski has proved the local uniformization theorem for v in projective form [9].

Now we generalize Theorem 2 (local simultaneous resolution for rational valuations) to higher varieties.

THEOREM 2A. *This is the same as Theorem 2 except let the dimension of K/k be arbitrary.*

Proof. In view of Proposition 3A and Theorem 13 the proof is the same as that of Theorem 2.

The results of Sections 7 and 8 (simultaneous resolvability or non-resolvability) and the partial generalizations of the results of Section 7 to higher varieties seems to be somewhat related to (and hence they might throw some light on) the possible approach to the problem of resolution of singu-

larities for higher varieties (over an algebraically closed ground field of characteristic zero) which has been proposed by Zariski in [13]. Hence before giving these generalizations of the results of Section 7, we wish to make a remark concerning this proposed method of Zariski. This method calls for proving the following stronger conjectural form of the resolution problem.

THEOREM Z. (Theoréme de Réduction of pages 4-5 of [13]). *If V_r is a pure r-dimensional variety regularly immersed in the direct product F_{r+1} of $r+1$ projective lines (over the field of complex numbers), then there exists an anti-regular transformation f of F_{r+1} such that $f(F_{r+1})$ is non-singular and such that all the singularities of the total transform $f[V_r]$ of V_r are 2-fold normal crossings (for the definition of an s-fold normal crossings see [1]; what we call a "2-fold normal crossing" is in Zariski's terminology simply a "normal crossing").*

We shall prove below that for any r there exists a V_r satisfying the conditions of Theorem Z such that if f is an anti-regular transformation of F_{r+1} with the properties: (1) $f(F_{r+1})$ is non-singular and (2) all the singularities of $f[V_r]$ are normal crossings; then $f[V_r]$ has a singularity which is an $(r+1)$-fold normal crossing.

Let $t=r+1$ and let $x_1, x_2, \cdots, x_t$ be t algebraically independent elements over an algebraically closed ground field k of characteristic zero. Let $y_i = x_i^p$ where p is a prime number. Let $K = k(y_1, y_2, \cdots, y_t)$ and let $K^* = k(x_1, x_2, \cdots, x_t)$. Let $q_1, q_2, \cdots, q_t$ be rationally independent positive real numbers. Now there is a unique zero dimensional valuation v^* of K^*/k such that $v^*(x_i) = q_i$ for $i = 1, 2, \cdots, t$. Let v be the K-restriction of v^*. Let G be the galois group of K^*/K. Then G is the direct product of t cyclic groups of order p and *hence G is not a direct product of s cyclic groups for any $s < t$.* It is clear that the ramification index of v^* over v is (pt) and hence G is also the splitting group of v^* over v.

In a natural way we can consider K/k to be the function field of the direct product F_t of t projective lines over k. Let F^* be a K^*-normalization of F_t and consider the branch locus on F_t of the transformation between F_t and F^*. By a Theorem of Zariski (Theorem 1 of [1]) this branch locus is a pure r-dimensional variety and we let it be our V_r. The conditions of immersion are readily adjusted (for instance: roughly speaking, take a line not on $y_1 y_2 \cdots y_t = 0$ as a factor line of F_t etc.). Let f be an anti-regular transformation of F_t satisfying conditions (1) and (2) above. Let $F' = f(F_t)$ and F'^* be a K^*-normalization of F'. Let D be the branch locus on F' of

the transformation between F' and F'^* and let $V' = f[V_r]$. Then $D \subset V'$ and hence all the singularities of D are normal crossings. Let P be the center of v on F' and let P^* be the center of v^* on F'^*. Then $G \supset G(P^*:P) \supset$ (the splitting group of v^* over v), see [3]. Therefore $G(P^*:P) = G$. Therefore P is on D and hence P is an s-fold normal crossing of D for some s. By Theorem 2 of [1], $s = t$. .Therefore P is a t-fold normal crossing also of V'. Q. E. D.

This shows that Theorem Z is not true in its present form. Also we have that, Theorem Z must be weakened at least to the point as to read: . . . every singularity of $f[v_r]$ is an s-fold normal crossing with $s \leqq r+1$ (s depending on the singularity).

Now we proceed to generalize Theorems 9 and 10 of Section 7 to higher varieties and we shall closely follow the proofs of these theorems.

THEOREM 9A. *Let K be an r-dimensional algebraic function field of characteristic $\neq 2$ and K^* a quadratic extension of K. Let V be a non-singular projective model of K and let V^* be K^*-normalization of V. Let D be the branch locus on V for the transformation between V and V^*. By Theorem 7 we have that D is pure $r-1$ dimensional. Assume that the only singularities of D are s-fold normal crossings with $s \leqq r$ (s depends on the singularity). Then there exists a non-singular model V_1 of K obtained from V by a sequence of monoidal transformations such that the K^*-normalization V_1^* of V_1 and the branch locus D_1 on V_1 are both non-singular.*

Proof. (We only give the additional considerations to supplement the proof of Theorem 9.) If $P \in D$ is an s-fold normal crossing of D then we shall denote the integer s by $s(P)$. Let $t = \max_{P \in D}[s(P)]$. If $t = 1$ there is nothing to prove, so assume that $t > 1$. Fix $P \in D$ with $s(P) = t$. Let (R, M) be the quotient ring of P on V and let $(x_1, x_2, \cdots, x_r)$ be a basis of M such that $x_1 x_2 \cdots x_t = 0$ is a local equation of D at P. There exists a primitive element z of K^*/K such that $z^2 = dx_1 x_2 \cdots x_t$ where d is a unit in R. Let v be a zero dimensional valuation of K/k having center P on V and let (R', M') be the quotient ring of the center P' of v on the variety V' obtained from V by a monoidal transformation centered at the $(r-2)$-dimensional subvariety W with local P-equation $x_{t-1} = 0 = x_t$. Let $y_i = x_i$ for $i = 1, 2, \cdots, t-2$. If $v(x_{t-1}) \neq v(x_t)$, let $y_{t-1} = x_t/x_{t-1}$ and $z_1 = z/x_{t-1}$ in case $v(x_{t-1}) < v(x_t)$, and let $y_{t-1} = x_{t-1}/x_t$ and $z_1 = z/x_t$ in case $v(x_t) < v(x_{t-1})$; then z_1 is a primitive element K^*/K with $z_1^2 = dy_1 y_2 \cdots y_{t-1}$ and $(y_1, y_2, \cdots, y_{t-1})$ is part of a minimal basis of M'. If $v(x_{t-1}) = v(x_t)$, then letting $z_1 = z/x_t$ we have that $z_1^2 = d_1 y_1 y_2 \cdots y_{t-2}$, where d_1 is a unit

in R' and $(y_1, y_2, \cdots, y_{t-2})$ is part of a minimal basis of M'. Thus in either case $s(P') < t$. Similar things happen to any other point on the subvariety W. Hence if there is still a point Q on V' with $s(Q) = t$ then Q does not lie on the total transform of W on V'. Now we treat Q on V' as we treated P on V and so on. Eventually we achieve a reduction in t. We continue till t reduces to 1.

THEOREM 10A. *Let the assumption be as in Theorem* 9A *except assume that K is of characteristic $\neq 3$, K^* is a cubic cyclic extension of K and that K contains a primitive cube root of unity. Then we have the same conclusion as in Theorem* 9A.

Proof. Combine the arguments used in the proofs of Theorems 10 and 9A.

COLUMBIA UNIVERSITY.

REFERENCES

[1] S. Abhyankar, "On the ramification of algebraic functions," *American Journal of Mathematics*, vol. 77 (1955), pp. 575-592.

[2] ———, "On the valuations centered in a local domain," *American Journal of Mathematics*, vol. 78 (1956), pp. 321-348.

[3] ———, "Local uniformization on algebraic surfaces over ground fields of charac-$p \neq 0$," *Annals of Mathematics*, vol. 63 (1956), pp. 491-526.

[4] ———, "On the field of definition of a nonsingular birational transform of an algebraic surface," *Annals of Mathematics* (to appear).

[5] Wolfgang Krull, "Galoissche Theorie bewerteter Körper," *Sitzungsberichte der Bayerischen Akademie der Wissenschaften*, München (1930), pp. 225-238.

[6] T. Matsusaka, "A note on my paper, 'Some theorems on abelian varieties'," *National Science Report of the Ochanomizu University*, vol. 5 (1954), pp. 21-23.

[7] Van der Waerden, *Modern Algebra*, vol. II, New York, 1950.

[8] O. Zariski, "The reduction of the singularities of an algebrais surface," *Annals of Mathematics*, vol. 40 (1939), pp. 639-689.

[9] ———, "Local uniformization on algebraic varieties," *ibid.*, vol. 41 (1941), pp. 852-896.

[10] ———, "Foundations of a general theory of birational correspondences," *Transactions of the American Mathematical Society*, vol. 53 (1943), pp. 490-542.

[11] ———, "Reduction of the singularities of algebraic three-dimensional varieties," *Annals of Mathematics*, vol. 45 (1944), pp. 472-542.

[12] ———, "Theory and applications of holomorphic function on algebraic varieties over arbitrary ground fields," *Memoirs of the American Mathematical Society*, No. 5, 1951.

[13] ———, "Le problème de la réduction des singularités d'une variété algébrique," *Bulletin des Sciences Mathématiques*, vol. 78 (1954), pp. 1-10.

[14] ———, "On minimal models of algebraic surfaces," forthcoming.

SOME ALGEBRAIC NUMBER THEORY ESTIMATES BASED ON THE DEDEKIND ETA-FUNCTION.*

By HARVEY COHN.

1. Introduction. Recent papers in this Journal [1], [2] attest to the continued interest in bounds on class number, regulator, etc., of algebraic fields. One special type, known as the *pure cubic* field (see Section 2 below) has the advantage that the zeta-function residue reduces to a finite expression involving the Dedekind eta-function. From this expression, by using the modular sub-diivsion, we can obtain the estimate (see Section 3 below).

$$h \log \epsilon = O(|d|^{\frac{1}{2}} \log |d| \log\log |d|), \tag{1.1}$$

where d is the field discriminant, h is the class number, and $\epsilon(>1)$ is the fundamental unit. The estimate (1.1) is a slight improvement over Landau's estimate [8], $O(|d|^{\frac{1}{2}} \log^2 |d|)$ under broader choice of field. One can obtain the further estimate (see Section 4 below).

$$h = O(|d|^{\frac{1}{2}} \log\log |d|) \tag{1.2}$$

which is a somewhat greater improvement over Landau's [7] $O(|d|^{\frac{1}{2}} \log^2 |d|)$. A more direct application of the technique would be bounds on certain modular invariants which are included for purposes of illustration. The main significance of this paper, however, is an indication of the direct usefulness of modular functions for cubic field estimates.

2. Notation and formulas. To discuss the pure cubic field $K_{a,b}$, let

a, b	be two relatively prime square-free positive integers, $(ab \neq 1)$,
$K_{a,b}$	be the field generated by the real root $(ab^2)^{\frac{1}{3}}$,
h	be the class number of $K_{a,b}$,
$\epsilon(>1)$	be the fundamental unit of $K_{a,b}$,
k	be ab (or $3ab$) according as $a^2 - b^2 \equiv 0$ (or $\not\equiv 0$) mod 9,
d	be $-3k^2$, the discriminant of $K_{a,b}$.

Then according to the famous formula of Dedekind ([3]; p. 228)

$$\epsilon^h = \Pi H(\omega_1)/\Pi H(\omega_0), \tag{2.11}$$

* Received March 21, 1956.

where

$$\mathrm{H}(\omega) = \eta(\omega)\eta(-\overline{\omega})[i(\overline{\omega}-\omega)]^{\frac{1}{2}}, \quad (\mathrm{Im}\,\omega > 0), \tag{2.12}$$

and the Dedekind η-function is given by

$$\eta(\omega) = \exp(\pi i\omega/12)\prod_{n=1}^{\infty}(1-\exp 2\pi i\omega n) \tag{2.13}$$

Here ω, $\overline{\omega}$ are complex conjugates and the variables ω_0 and ω_1 range over roots of quadratic forms of discriminant $-3k^2$ arranged in accordance with certain reciprocity properties. The detailed description of the choice of ω_0 and ω_1 ([3]; p. 205) is unnecessary in what follows. (Indeed we permit the worst possible arrangement!) It will suffice to say that the numbers ω_0 and ω_1 each range over one-ninth of the roots ω formed with integral x, y, z:

$$\omega = (x+z\rho)/y, \qquad (\rho = (-1+i3^{\frac{1}{2}})/2), \tag{2.2}$$

where $x \geqq 0$, $y > 0$, $z > 0$, and $zy = k$, $0 \leqq x < y$ while y and $x + z\rho$ are relatively prime (in the integral domain of ρ). There are $9k''$ such roots, where from the factorization

$$k = 3^e \prod p, \qquad 0 \leqq e \leqq 2, \qquad p \text{ prime} \neq 3,$$

$$9k'' = 3^e \prod (p - (3/p)), \tag{2.3}$$

for the quadratic residue symbol $(3/p)$.

The formula (2.11) is of course reminiscent of Dirichlet's corresponding formula for the real quadratic field [6], with some very favorable differences. First of all, in the quadratic case the *number* of factors has the rough order of magnitude of the discriminant, while here it has the smaller order of magnitude of k'', or roughly the square root of the discriminant. More important, here the *size* of the factors $\mathrm{H}(\omega)$ permits a significant estimate from above or below by virtue of the fundamental domain of the modular group ([4]; p. 14).

Specifically each ω can be replaced by some equivalent value Ω for which

$$\Omega = (s'\omega - r')/(-s\omega + r), \qquad -r's + s'r = 1 \tag{2.41}$$

(for some positive or negative integers r, s, r', s') such that

$$|\,\mathrm{Re}\,\Omega\,| \leqq \tfrac{1}{2}, \qquad |\,\Omega\,| \geqq 1, \qquad \mathrm{Im}\,\Omega > 0, \tag{2.42}$$

and for which $\mathrm{H}(\Omega) = \mathrm{H}(\omega)$. The (uniquely determined) value of $\mathrm{Im}\,\Omega$ is called the *valence* of ω (or Ω) and denoted by

$$\mathrm{Im}\,\Omega = V(\omega). \tag{2.43}$$

From the shape of the fundamental domain (2.42),

$$V(\omega) \geqq 3^{\frac{1}{2}}/2. \tag{2.44}$$

Thus $\log|\eta(\Omega)|$ is approximated to within a bounded error by $\pi V(\omega)/12$, yielding, (after Dedekind [3]; p. 231),

$$|\log H(\omega) - \tfrac{1}{2}\log 2V(\omega) + \pi V(\omega)/6| \leqq \gamma_0, \tag{2.51}$$

and, easily,

$$|\log H(\omega)| \leqq \gamma_1 V(\omega), \tag{2.52}$$

where the γ_i will refer to positive absolute constants.

Now the estimation problem reduces easily from (2.11), and (2.52) to

$$h \log \epsilon \leqq \gamma_2(\sum V(\omega_1) + \sum V(\omega_0)), \tag{2.61}$$

and finally to

$$h \log \epsilon \leqq \gamma_2 \sum_{zy=k} \sum_{x=0} V([x+z\rho]/y), \tag{2.62}$$

dropping the restriction that y and $x+x_\rho$ be relatively prime.

3. The $h \log \epsilon$ estimate.[1] We consider all the individual arguments of the "valence sum" (2.62), namely,

$$\psi = (x+z\rho)/y = x/y + \alpha + \beta i. \tag{3.11}$$

with the abbreviations

$$\alpha = z/(2y), \qquad \beta = 3^{\frac{1}{2}}z/(2y). \tag{3.12}$$

Consider a unimodular transformation in integral coefficients

$$\Psi = (s'\psi - r')/(-s\psi + r), \qquad rs' - sr' = 1, \tag{3.21}$$

for which Ψ lies in the fundamental domain (2.42). Hence,

$$3^{\frac{1}{2}}/2 \leqq \operatorname{Im}\Psi = \beta\,|\,s(x/y+\alpha) - r + \beta i\,|^{-2} = V(\psi). \tag{3.31}$$

Therefore,

$$s^2 \leqq 2 \cdot 3^{-\frac{1}{2}}/\beta, \tag{3.32}$$

$$[r - s(x/y+\alpha)]^2 \leqq 2\beta/3^{\frac{1}{2}}. \tag{3.33}$$

when $s=0$ and (say) $r=1$, $V(\psi)=\beta$.

We finally rearrange the terms in valence sum (2.62) according to the

[1] The author is indebted to Professor A. Weil for suggestions leading to the simplification of the estimates.

fraction r/s (1/0 included). First of all $s=0$ only if $\beta \geqq 3^{\frac{1}{2}}/2$ or if $z \geqq y$. For such values of ψ, the valence sum is majorized by

$$(3.4) \qquad S_0 = \sum_{s=0} V([x+z\rho]/y) = (3^{\frac{1}{2}}/2) \sum_{zy=k} \sum_{x=0}^{y-1} (z/y) = 3^{\frac{1}{2}}\sigma(k)/2,$$

where $\sigma(k)$ represents the sum of the positive divisors of k.

Next we consider the portion of the valence sum restricted to fractions r/s with $s \neq 0$ and (say) positive. The contribution is written as

$$(3.51) \qquad S_1 = \sum_{zy=k}{}' \sum_{x=0}^{y-1}{}' \beta/\{[s(x/y+\alpha)-r]^2 + s^2\beta^2\},$$

where α, β always refer to abbreviations (3.12) and r/s is regarded as a function of x, y, and z (chosen earlier in equation (3.31)). The primes indicate that the sum is restricted so that $s>0$.

The inside sum (formed by fixing y and z) can be estimated by further fixing s and regarding x as the variable with r dependent on x. The sum S_1 is amply majorized if we let x go from $-\infty$ to $+\infty$ for *each* r *that occurs*. We then can use the estimate

$$(3.52) \qquad \sum_{x=-\infty}^{\infty} \kappa/[\zeta^2 + (x-\xi)^2] \leqq \kappa\zeta^{-2} + \kappa \int_{-\infty}^{\infty} dx/(\xi^2+x^2) = (\zeta^{-2} + \pi\zeta^{-1})\kappa$$

where $\zeta = \beta y$, $\xi = (ry/s) - y\alpha$, and $\kappa = \beta y^2/s^2$ in our context. In this manner the inside sum becomes transformed to a sum over *all* r/s *occurring* for a fixed y and z. The range of s is limited by inequality (3.32) to

$$(3.53) \qquad 1 \leqq s \leqq \mu, \qquad \mu^2 = 2\cdot 3^{-\frac{1}{2}}/\beta = 4/3 \cdot y/z,$$

and for each s, the range of numerators r is limited by $4s$, since from inequality (3.33) combined with

$$(3.54) \qquad 2\cdot 3^{-\frac{1}{2}} \geqq \beta = 3^{\frac{1}{2}}\alpha, \qquad 0 \leqq x < y,$$

we can conclude r lies in the interval $(-2/3^{\frac{1}{2}}, 5s/3 + 2/3^{\frac{1}{2}})$. We note finally from inequality (3.53), $2y \geqq 3z$, hence S_1 is majorized as follows:

$$(3.55) \qquad S_1 \leqq \sum_{\substack{zy=k \\ 2y \geqq 3z}} \sum_{s=1}^{\mu} 4s\beta y^2[\beta^{-2}y^{-2} + \pi/(\beta y)]/s^2,$$

$$(3.56) \quad S_1 \leqq \sum_{\substack{zy=k \\ 2y \geqq 3z}} [2y/(3^{\frac{1}{2}}z) + \pi y] \sum_{1}^{\mu} 4/s, \qquad (3.57) \quad S_1 \leqq \sum_{y|k} \gamma_3 y \log(4k/3),$$

since $\mu^2 = 4y/(3z) < 4k/3$. Hence the estimates (3.4) and (3.57) take us to

$$(3.6) \qquad h \log \epsilon \leqq \gamma_4 \sigma(k) \log k.$$

By the well known estimate [5], $\sigma(k) = O(k \log\log k)$, we obtain the result (1.1).

Perhaps a more concise way of presenting the basic result is to say that the average of the $9k''$ valences, $V((x+z\rho)/y)$, in (2.2), is $O(\log k)^{1+o(1)}$ as $k \to \infty$.

4. The class number estimate. To obtain an estimate for h from inequality (3.6) we consider the relation between ϵ and k using the following technique, (which must be presumed to be well-known although no specific reference comes to mind): Let $K'_{a,b}$ and $\bar{K}'_{a,b}$ be the conjugate fields to $K_{a,b}$ with ϵ' and $\bar{\epsilon}'$ corresponding to ϵ. Then since ϵ is a unit $|\epsilon'| = |\bar{\epsilon}'| = \epsilon^{-\frac{1}{2}}$, and Δ, the root-discriminant of ϵ, satisfies the inequality

$$(4.1) \qquad |\Delta| = |(\epsilon-\epsilon')^2(\epsilon-\bar{\epsilon}')^2(\epsilon'-\bar{\epsilon}')^2| \leqq (\epsilon+\epsilon^{-\frac{1}{2}})^4(2\epsilon^{-\frac{1}{2}})^2 \leqq \gamma_5\epsilon^3.$$

But $|\Delta| \geqq |d| = 3k^2$. Thus

$$(4.2) \qquad k \leqq \gamma_6\epsilon^{\frac{3}{2}},$$

which, together with estimate (3.6), yeilds

$$(4.3) \qquad h = O(\sigma(k))$$

and ultimately estimate (1.2). In both estimates (1.1) and (1.2) numerical constants could be easily supplied.

5. Norm estimate of modular invariants. A direct application of the sums (2.62) can be made to algebraic numbers of the type $j(\omega)$, where $j(z) = 1728J(z)$, for $J(z)$ the Klein modular function and ω a "positive-imaginary" quadratic number. For instance the quantities ω described in (2.2) are such that the $j(\omega)$ are a set of algebraic integers containing the conjugates of each element ([4]; p. 74). Thus taking one element, say $j(k\rho)$, we find

$$(5.1) \qquad \log|N(j(k\rho))| \leqq \gamma_7 \sum_{zy=k} \sum_{x=0}^{y-1} V([x+z\rho]/y)$$

and, as before,

$$(5.2) \qquad |N(j(k\rho))| \leqq \exp(\gamma_8\sigma(k)\log k)$$

where $N(\cdots)$ represents the (absolute) norm of an algebraic number.

In a more general situation the quadratic field will not have class number unity. Thus if λ is a positive-imaginary quadratic number, the

conjugates of $j(k\lambda)$ will be chosen from a finite number of sets of type (2.2), yielding ([4]; p. 76)

$$|N(j(k\lambda))| \leqq \exp(c(\lambda) k \log k \log\log k), \tag{5.3}$$

where $c(\lambda)$ is a constant depending on λ (and not on k).

NATIONAL BUREAU OF STANDARDS
AND
WASHINGTON UNIVERSITY.

REFERENCES.

[1] N. C. Ankeny, R. Brauer, and S. Chowla, "A note on the class numbers of algebraic number fields," *American Journal of Mathematics*, vol. 78 (1956), pp. 51-61.

[2] R. Brauer, "On the zeta-functions of algebraic number fields," *American Journal of Mathematics*, vol. 69 (1947), pp. 243-250.

[3] R. Dedekind, *Gesammelte mathematische Werke*, vol. II, Braunschweig, 1931.

[4] R. Fueter, *Vorlesungen uber die singulären Moduln und die komplexe Multiplikation der elliptischen Funktionen*, vol. I, Leipzig, 1924.

[5] G. H. Hardy, *An introduction to the theory of numbers*, Oxford, 1938, p. 264.

[6] H. Hasse, *Vorlesungen über Zahlentheorie, Berlin*, 1950, p. 396.

[7] E. Landau, "Abschätzung von Charaktersummen, Einheiten, und Klassenzahlen," *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, 1918, p. 95.

[8] ———, "Verallgemeinerung eines Pólyaschen Satzes auf algebraische Zahlkörper," *ibid.*, 1918, p. 480.

FINITE FANO PLANES.*

By ANDREW M. GLEASON.

Introduction.[1] By a projective plane $\mathcal{P}$ we will mean a set $\mathcal{P}$ of points, certain subsets of which are called lines, such that the usual incidence axioms hold; viz.

I. If p and q are distinct points, there is a line L such that $p \in L$ and $q \in L$.

II. If L and M are distinct lines then $L \cap M$ contains exactly one point.

We shall also assume an axiom of non-triviality:

III. There exist four distinct points no three of which are members of the same straight line.

We definitely do not assume any further configuration axiom such as Desargues' without explicit mention. Although the notation is not symmetric, the axioms are well-known to be self-dual, so that any theorem about points and lines can be immediately reasserted after interchanging these concepts.

The cardinal of any two lines can easily be shown to be the same and this cardinal is the same as that of the set of lines containing a given point. If this cardinal is finite, say $n+1$, then n is called the order of the projective plane. In this case there will be n^2+n+1 points altogether and the same number of lines.

For any field F, or even a skew field, a model of this axiom system can be obtained by considering a three dimensional vector space over F, letting a "point" be a one-dimensional subspace and a "line" be a set of "points" which are subspaces of some two-dimensional subspace. In this model the classical configuration of Desargues will hold. Conversely, any projective plane in which Desargues' theorem is valid is isomorphic to one obtained by the preceding construction. One finds thus a one-to-one correspondence between isomorphism classes of Desarguesian projective planes and isomorphism classes of skew fields.

---

* Received February 15, 1956.

[1] The author is indebted to R. Baer, P. Dembowski, M. Hall, Jr., W. Pierce, and the referee for assistance in preparing the introduction and bibliography.

The analysis of projective planes with the aid of axioms I, II, and III only has not been very illuminating. A number of examples, some finite, have been constructed of projective planes in which Desargues' theorem fails. The finite ones have all been obtained by making systematic changes in the incidence structure of a Desarguesian plane; these changes do not affect the order of the plane which remains, in all known cases, a prime power. Whether this is a happenstance of our ignorance or a theorem has yet to be decided. Bruck and Ryser [2], by an ingenious number theoretic argument, have excluded an infinite list of integers as possible orders, and direct combinatorial study (finally using a giant computer[2]) has shown that only Desarguesian planes occur with orders less than 9.

More satisfactory results have been obtained by strengthening I, II, and III with other configuration theorems. Miss Moufang [8], for example, assumed that harmonic conjugates are uniquely determined by the usual construction; then, assuming also that no point is self-conjugate, she introduced affine coordinates from an alternative division ring. The small theorem of Desargues, taken as an axiom, also leads to coordinates from an alternative division ring (Hall [3]). Since a finite alternative division ring is necessarily a field (Zorn [13]), we can conclude that a finite projective plane satisfying the small theorem of Desargues is unrestrictedly Desarguesian.

The extra proviso of Miss Moufang that no point be self-conjugate leaves a gap in our knowledge. Assuming the unicity of the construction, if self-conjugacy occurs once it is universal [10]; this leads us to the study of planes in which this occurrence is postulated, which we call Fano planes. They are characterized by the fact that the diagonal points of any quadrangle are collinear.[3] Our main result is that a finite Fano plane is Desarguesian.

Assumptions on the nature of the collineation (automorphism) group are another way to supplement axioms I, II, and III. For a Desarguesian projective plane of order $n = p^m$ (where p is prime) the collineation group has order $mn^3(n-1)^2(n^2+n+1)(n+1)$. One expects, of course, that a non-Desarguesian plane will have far fewer collineations, and this is indeed the case so far as known. The collineation group of a Desarguesian plane always contains a cyclic transitive subgroup [12]; Hall [4] and Mann and Evans [7] have investigated the converse and have shown that, for $n \leqq 1600$, the existence of a cyclic transitive group of collineations in a plane of order n

[2] Marshall Hall, personal communication.

[3] As remarked by the referee, there is a good deal to be said for calling such a plane an Anti-Fano plane, since the non-existence of a quadrilateral with collinear diagonal points was taken as an axiom by Fano.

implies that n is a prime power. In this paper we shall show that the existence in a finite plane of a relatively small number of collineations of a special type is sufficient to establish the small, and hence the full, theorem of Desargues; then we show that a finite Fano plane has sufficiently many of these special collineations.

Since our method makes particular use of the small Reidemeister configuration (called configuration G) it is closely related to the work of Klingenberg [5] who shows that the large Reidemeister configuration implies the theorem of Desargues.

For the sake of completeness we have included proofs of a number of already known results (see [9] for a comprehensive treatment of the subject), so that the paper is self-contained beyond the results of Hall just quoted on the small theorem of Desargues and the group theoretic interpretation of the small theorem given in Lemma 1.4.

1. Collineations. Consider a projective plane $\mathfrak{P}$ satisfying axioms I, II, and III. A collineation of $\mathfrak{P}$ is a permutation α of the set $\mathfrak{P}$ such that $\alpha(x)$, $\alpha(y)$, and $\alpha(z)$ are collinear if and only if x, y, and z are collinear. Evidently the collineations form a group, Π. Any collineation of $\mathfrak{P}$ induces a permutation of the set $\mathfrak{L}$ of lines of $\mathfrak{P}$, and this permutation preserves the phenomenon of concurrence. This new permutation representation of Π is evidently faithful. If every point of a certain line is left fixed by a collineation, we say the line is left fixed point-wise. Similarly, if every line through a point is fixed, we say that the point is left fixed line-wise.

For a given point x and line L consider the group G_{xL} of all collineations which leave x fixed line-wise and L fixed point-wise. In general we must expect that this group is trivial; that is, contains only the identity. We shall analyse the opposite possibility.

LEMMA 1.1. *If $\alpha \in G_{xL}$ and α leaves fixed a point y other than x and not on L (or, if α leaves fixed a line other than L and not passing through x), then α is the identity.*

Proof. Suppose that α leaves fixed y where $y \notin L$ and $y \neq x$. Then α leaves fixed every line M through y since y and $M \cap L$ are both fixed points. Therefore α leaves fixed every point which is the intersection of a line through x with a line through y. This includes all the points of $\mathfrak{P}$ except those on the line joining x to y. Hence every line contains at least two fixed points and is itself fixed. Finally every point is fixed, being the intersection of two fixed lines. The other hypothesis is dual to the first.

LEMMA 1.2. *The set* $G_L = \bigcup_{x \in L} G_{xL}$ *is a group.* G_L *consists of the identity and those collineations of* $\mathcal{P}$ *which leave* L *fixed pointwise, but have no other fixed points. If, for two distinct points* x *and* y *of* L, G_{xL} *and* G_{yL} *are non-trivial, then* G_L *is abelian; furthermore, every element of* G_L *(except the identity) has the same order, either infinity or a finite prime.*

Proof. By Lemma 1.1 any member of G_L fulfills the description given in the second statement. Conversely, consider a collineation α which leaves L pointwise fixed and has no other fixed points. For any $p \notin L$, let M be the line joining p and $\alpha(p)$; say M meets L at x. Now $\alpha(M)$ must contain both $\alpha(p)$ and $\alpha(x) = x$; so $\alpha(M) = M$ and M is fixed. Thus every point is on a fixed line. Any two of these fixed lines must meet at a fixed point which is therefore on L, so it must be x. Now we can see immediately that every line through x is fixed, so $\alpha \in G_{xL} \subset G_L$. This proves the second statement of the lemma.

Let α and β be any two elements of G_L, say $\alpha \in G_{xL}$ and $\beta \in G_{yL}$. To prove that G_L is a group, we must prove that $\alpha\beta^{-1}$ is in G_L. We may assume that neither α nor β is the identity and that $x \neq y$, since in these cases, $\alpha\beta^{-1} \in G_L$ trivially. Obviously, $\alpha\beta^{-1}$ is a collineation which leaves L pointwise fixed, so it will be sufficient to show that $\alpha\beta^{-1}$ has no fixed point not on L. Let p be any point not on L; set $q = \beta^{-1}(p)$ and $r = \alpha(q)$. Now $q, r = \alpha(q)$, and x are collinear and also $q, p = \beta(q)$, and y; since these two lines are distinct and meet at q we cannot have $p = \alpha\beta^{-1}(p) = r$ unless $p = q = r$; in this case, α and β would both have a fixed point not on L and would be the identity by 1.1. This case has been excluded.

We next prove that if x and y are distinct points of L, $\alpha \in G_{xL}$, and $\beta \in G_{yL}$, then α and β commute. To this end we must show that $\alpha\beta(p) = \beta\alpha(p)$ for any point p; this being trivial for $p \in L$, we assume the contrary. The following triples are collinear

$$\begin{array}{ll} x, p, \alpha(p) & y, p, \beta(p) \\ x, \beta(p), \alpha\beta(p) & y, \alpha(p), \beta\alpha(p) \\ x, \beta(p), \beta\alpha(p) & y, \alpha(p), \alpha\beta(p). \end{array}$$

(On the left, the first two derive from the fact that α leaves x fixed linewise; the third, is obtained from the first by applying β. Similarly on the right). These triples characterize both $\alpha\beta(p)$ and $\beta\alpha(p)$ as the point of intersection of the lines joining x to $\beta(p)$ and y to $\alpha(p)$. This proves that $\alpha\beta = \beta\alpha$.

Let us observe that if $x \neq y$, $G_{xL} \cap G_{yL} = \{\epsilon\}$, because any common

collineation satisfies the second hypothesis of Lemma 1.1. For the remainder of the proof we assume that at least two of the groups G_{xL} are non-trivial, which is equivalent to assuming that G_L contains each of the groups G_{xL} properly.

To prove that G_L is abelian, there is left only the case that α and β both come from the same subgroup G_{xL}. Choose a collineation $\gamma \in G_L - G_{xL}$; then certainly $\beta\gamma \notin G_{xL}$. By our previous result, α commutes with both γ and $\beta\gamma$; therefore, with β.

To demonstrate the last statement of the lemma, we suppose that G_L contains an element (other than ϵ) of finite order; this element can be chosen to have prime order p; let it be α and say $\alpha \in G_{xL}$. For any $\beta \notin G_{xL}$ say $\beta \in G_{yL}$ and $\alpha\beta \in G_{zL}$; here $y \neq z$. Now $\beta^p \in G_{yL}$ and also $\beta^p = \alpha^p\beta^p = (\alpha\beta)^p \in G_{zL}$; so $\beta^p = \epsilon$. Thus every element of $G_L - G_{xL}$ has order p; repeating the argument starting with β, we see that every element of G_L (except ϵ) has order p. This finishes the proof of Lemma 1.2.

LEMMA 1.3. *Suppose that $\mathfrak{P}$ is finite. If L is a line and x and y are two distinct points of L such that neither G_{xL} nor G_{yL} is trivial, then every element of either group (except ϵ) has the same prime order. If x is a point and L and M are two distinct lines through x such that neither G_{xL} nor G_{xM} is trivial, then every element of either group (except ϵ) has the same prime order.*

Proof. It is clear that, if $\mathfrak{P}$ is finite, G_L must be finite, so the first statement is part of the last lemma. The second statement is dual to the first. In applying duality, G_{xL} and G_{xM} must be interpreted as groups of permutations of the set $\mathfrak{L}$ of lines; but, since the statement concerns their abstract structure, it is immaterial.

LEMMA 1.4. *A necessary and sufficient condition that G_L be transitive on the points of $\mathfrak{P} - L$ is that the small theorem of Desargues hold on L.*

This has frequently been used in treatments of the coordinatization problem, so we omit the proof, which may be found, for example, in [1].

LEMMA 1.5. *Suppose that $\mathfrak{P}$ is finite of order n. The cardinal of G_{xL} $(x \in L)$ divides n and the cardinal of G_L divides n^2. G_L is transitive on $\mathfrak{P} - L$ if and only if the cardinal of G_L is exactly n^2.*

Proof. Let M be some line through x other than L. G_{xL} can be regarded as a group of permutations of the n points of $M - \{x\}$. Since no element of G_{xL} except the identity leaves fixed any point of $M - \{x\}$, each primitive

subset has the same cardinal as $G_{x\,L}$, and therefore this cardinal divides n. The other assertions follow in the same way when G_L is regarded as a group of permutations of the n^2 points of $\mathfrak{P} - L$.

LEMMA 1.6. *Suppose that $\mathfrak{P}$ is finite of order n and that, for some line L, all of the groups $G_{x\,L}$ $(x \in L)$ have the same cardinal $h > 1$. Then the small theorem of Desargues holds on L.*

Proof. Let g be the cardinal of G_L. We know that g divides n^2, say $gm = n^2$. In view of the last two lemmas we need only prove that $m = 1$. Now G_L is the union of $n+1$ groups $G_{x\,L}$, each of cardinal h, with each pair having only the identity in common; hence, $(h-1)(n+1)+1 = g$, or $((h-1)(n+1)+1)m = gm = n^2$. This equation shows that $m < n$ (since $h > 1$) and $m \equiv 1 \pmod{n+1}$. Therefore, $m = 1$ and the lemma is proved.

LEMMA 1.7. *Suppose that H is a group of permutations of a finite set L and suppose that, for some prime p and each $x \in L$, there exists an element of H of order p which leaves x fixed but has no other fixed points. Then H is transitive.*

Proof. Let L_1 be a primitive subset of L. Choose $x \in L_1$, and let α be an element of H of order p which leaves x fixed but has no other fixed points. L_1 consists of x and a number of disjoint α-primitive blocks each of cardinal p; hence, cardinal $L_1 \equiv 1 \pmod{p}$. Now suppose that H is not transitive, so that we can choose $y \in L - L_1$. Let β be an element of H which has order p and the unique fixed point y. Now L_1 is a union of disjoint β-primitive blocks each of cardinal p, hence cardinal $L_1 \equiv 0 \pmod{p}$. This contradiction establishes the lemma.

THEOREM 1.8. *Let $\mathfrak{P}$ be a finite projective plane. Suppose, for every point x and line L with $x \in L$, that $G_{x\,L}$ is non-trivial. Then $\mathfrak{P}$ is Desarguesian.*

Proof. Lemma 1.3 implies immediately that every non-trivial element of any of the groups $G_{y\,M}$ $(y \in M)$ has the same prime order, say p.

Choose any line L and consider the group H_L of all collineations of $\mathfrak{P}$ which leave L fixed (not necessarily point-wise). H_L is represented (not faithfully) as a group of permutations of L. Now H_L contains as subgroups all of the groups $G_{x\,M}$ where $x \in L$ and M is any line through x and a non-trivial element of $G_{x\,M}$ $(M \neq L)$ has order p and acts on L with no fixed points except x itself; therefore, Lemma 1.7 applies and H_L is transitive on L. From the elementary formula $\alpha G_{x\,L}\alpha^{-1} = G_{\alpha(x)\,L}$ for $\alpha \in H_L$, it

now follows that all of the groups G_{xL} with $x \in L$ are conjugate in H_L. Now by Lemma 1.6, the small theorem of Desargues holds on the arbitrarily chosen line L. But a finite plane in which the small Desargues' theorem is valid is Desarguesian.

2. Projections. Consider a finite plane $\mathcal{P}$ of order n, a definite line L of $\mathcal{P}$ and a point x on L; we will obtain a sufficient condition that G_{xL} be non-trivial. Let $M_1, M_2, \cdots, M_n$ be an enumeration of the lines through x other than L. For any point a of $L - \{x\}$, let $\xi^a{}_{ji}$ denote the projection of M_i onto M_j from a; $\xi^a{}_{ji}$ is characterized by the fact that, for any $y \in M_i$, $\xi^a{}_{ji}(y) \in M_j$ and a, y, and $\xi^a{}_{ji}(y)$ are collinear. Obviously, $\xi^a{}_{kj}\xi^a{}_{ji} = \xi^a{}_{ki}$. If b is also a point of $L - \{x\}$, then $\xi^a{}_{ij}\xi^b{}_{ji}$ is a permutation of M_i. From the incidence axioms we deduce immediately that, for fixed a and b $(a \neq b)$, $J^{ab}{}_i = \{\xi^a{}_{ij}\xi^b{}_{ji} \mid j = 1, 2, \cdots, n\}$ is a simply transitive family of permutations of $M_i - \{x\}$.

LEMMA 2.1. *There is a one-to-one correspondence between elements of G_{xL} and permutations of M_i which commute with every permutation of the form $\xi^a{}_{ij}\xi^b{}_{ji}$. The correspondence is given by restricting elements of G_{xL} to M_i.*

Proof. Any element α of G_{xL} leaves fixed the line M_i; consequently, the restriction of α to M_i is a permutation of M_i. Obviously, the restriction map is a homomorphism of G_{xL}. Since no element of G_{xL} (except ϵ) can have any fixed points not on L, the restriction map is one-to-one.

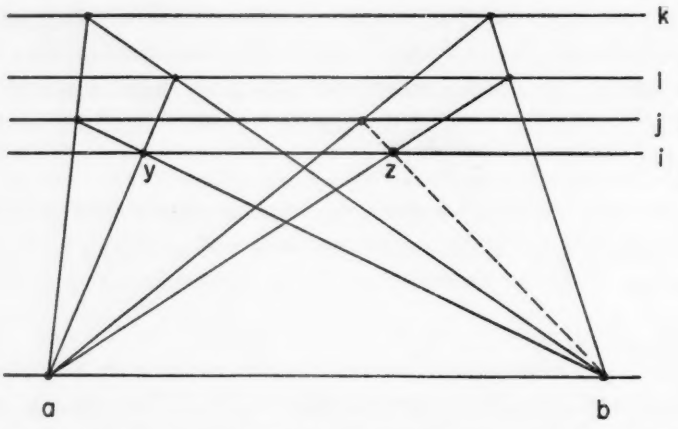
Suppose that $\alpha \in G_{xL}$, $y \in M_i$, $z \in M_j$, and $\xi^a{}_{ji}(y) = z$. We have $\alpha(a) = a$, $\alpha(y) \in M_i$, and $\alpha(z) \in M_j$ by the definition of G_{xL}; a, y, and z are collinear, so a, $\alpha(y)$, and $\alpha(z)$ are also; whence $\xi^a{}_{ji}\alpha(y) = \alpha(z) = \alpha\xi^a{}_{ji}(y)$. This being valid for any y on M_i, $\xi^a{}_{ji}\alpha = \alpha\xi^a{}_{ji}$. Then clearly, $\xi^a{}_{ij}\xi^b{}_{ji}\alpha = \alpha\xi^a{}_{ij}\xi^b{}_{ji}$; therefore α and its restriction to M_i commute with all permutations of the form $\xi^a{}_{ij}\xi^b{}_{ji}$.

Now suppose β is a permutation of M_i which commutes with all permutations of the form $\xi^a{}_{ij}\xi^b{}_{ji}$. If $a \neq b$ and $i \neq j$, the only fixed point of $\xi^a{}_{ij}\xi^b{}_{ji}$ is x, so β must leave x fixed; hence, we may define a permutation α of $\mathcal{P}$ as follows: Pick a point a on $L - \{x\}$ and let $\alpha(y) = \xi^a{}_{ji}\beta\xi^a{}_{ij}(y)$ if $y \in M_j$ and $\alpha(y) = y$ if $y \in L$. We shall see that α is a collineation. To prove this it suffices to consider collinear points b, y, and z, where $b \in L$, $y \in M_j$, and $z \in M_k$, and show that b, $\alpha(y)$, and $\alpha(z)$ are collinear. We have $z = \xi^b{}_{kj}(y)$ and

$$\begin{aligned}\xi^b{}_{kj}\alpha(y) &= \xi^b{}_{kj}\xi^a{}_{ji}\beta\xi^a{}_{ij}(y) = \xi^a{}_{ki}(\xi^a{}_{ik}\xi^b{}_{ki})(\xi^b{}_{ij}\xi^a{}_{ji})\beta\xi^a{}_{ij}(y) \\ &= \xi^a{}_{ki}\beta(\xi^a{}_{ik}\xi^b{}_{ki})(\xi^b{}_{ij}\xi^a{}_{ji})\xi^a{}_{ij}(y) = (\xi^a{}_{ki}\beta\xi^a{}_{ik})\xi^b{}_{kj}(y) = \alpha(z);\end{aligned}$$

that is, b, $\alpha(y)$, and $\alpha(z)$ are collinear. It is clear that α leaves L fixed point-wise and x fixed line-wise; i.e., $\alpha \in G_{xL}$. This completes the proof of the lemma.

We will now assume that, for any a and b of $L - \{x\}$ and any indices i, j, k, l, if $\xi^b{}_{ij}\xi^a{}_{jk}\xi^b{}_{kl}\xi^a{}_{li}$ (which is a permutation of M_i) has a fixed point, then it is the identity. This corresponds to the closure of the following figure. Here we have taken x at infinity. The diagram shows y as a fixed point of the fourfold projection and the closure of the figure shows that another point z is also left fixed. The figure consists of 13 lines and 11 points



(counting x); five lines go through each of the three points a, b, and x, and three through every other point. (There is a degenerate case if M_j coincides with M_l or M_i with M_k. The closure of the degenerate case is implied by the closure of all non-degenerate cases.) This configuration will be referred to as configuration G because it implies that the set $J^{ab}{}_i$ is a group. We shall say that the configuration G holds in $\mathfrak{P}$ if it holds regardless of the choice of x and L. It is easy to show that configuration G holds in any Desarguesian plane. This configuration has been studied by Reidemeister [11] who obtains the following lemma.

LEMMA 2.2. *If configuration G holds in $\mathfrak{P}$ then the set $J^{ab}{}_i$ is a group. Conversely, if $J^{ab}{}_i$ is a group for all choices of L, x, a, b, and i, then configuration G holds in $\mathfrak{P}$.*

Proof. Suppose that $\xi^a{}_{ik}\xi^b{}_{ki}$ and $\xi^a{}_{il}\xi^b{}_{li}$ are two given members of $J^{ab}{}_i$. Choose any point y of M_i and an index j such that $(\xi^a{}_{ik}\xi^b{}_{ki})(\xi^a{}_{il}\xi^b{}_{li})^{-1}(y) = \xi^a{}_{ij}\xi^b{}_{ji}(y)$. This is possible since $J^{ab}{}_i$ is a transitive family. Then y is a fixed point of $\xi^b{}_{ij}\xi^a{}_{ji}\xi^a{}_{ik}\xi^b{}_{ki}\xi^b{}_{il}\xi^a{}_{li} = \xi^b{}_{ij}\xi^a{}_{jk}\xi^b{}_{kl}\xi^a{}_{li}$; hence this permutation

has only fixed points. It follows immediately that $(\xi^a{}_{ik}\xi^b{}_{ki})(\xi^a{}_{il}\xi^b{}_{li})^{-1} = \xi^a{}_{ij}\xi^b{}_{ji}$; thus $J^{ab}{}_i$ is a group.

Conversely, assume that $J^{ab}{}_i$ is always a group. If $\xi^b{}_{ij}\xi^a{}_{jk}\xi^b{}_{ki}\xi^a{}_{il} = (\xi^a{}_{ij}\xi^b{}_{ji})^{-1}(\xi^a{}_{ik}\xi^b{}_{ki})(\xi^a{}_{il}\xi^b{}_{li})^{-1}$ has a fixed point, it is the identity, being an element of $J^{ab}{}_i$ which is always a simply transitive family of permutations.

LEMMA 2.3. *Suppose that A, B, and C are subgroups of a group and suppose that the elements of A and B can be so indexed that, for each i, $\alpha_i\beta_i \in C$. Assume, moreover, that every element of C can be written in the form $\alpha_i\beta_i$. Then A and B commute as complexes; i.e., $AB = BA$.*

Proof. Given α_i and β_j, choose k so that $(\alpha_j\beta_j)(\alpha_i\beta_i) = \alpha_k\beta_k$; then $\beta_j\alpha_i = (\alpha_j^{-1}\alpha_k)(\beta_k\beta_i^{-1})$, proving $BA \subset AB$. Taking inverses, the latter relation becomes $AB \subset BA$, whence $AB = BA$.

LEMMA 2.4. *If $A_1, A_2, \cdots, A_n$ is a sequence of subgroups of a group which mutually commute as complexes, then $B = A_1A_2\cdots A_n$ is a group. Moreover, if the groups A_i are finite, then B is finite and every prime factor of the cardinal of B is a factor of the cardinal of A_i, for some i.*

Proof. This is well-known for $n = 2$. The general case follows easily by induction.

THEOREM 2.5. *If configuration G holds in $\mathfrak{P}$ and the order of $\mathfrak{P}$ is a prime power, then $\mathfrak{P}$ is Desarguesian.*

Proof. Let L be a fixed line of $\mathfrak{P}$ and x, a, b, and c points of L. Observe that $J^{ab}{}_i = J^{ba}{}_i$ because the latter set is made up of the inverses of the members of the former, which is a group. Enumerating $J^{ab}{}_i$ in the order $\xi^b{}_{ij}\xi^a{}_{ji}$, $j = 1, 2, \cdots, n$ and $J^{ac}{}_i$, $\xi^a{}_{ij}\xi^c{}_{ji}$, the corresponding products $\xi^b{}_{ij}\xi^a{}_{ji}\xi^a{}_{ij}\xi^c{}_{ji} = \xi^b{}_{ij}\xi^c{}_{ji}$ run through the group $J^{bc}{}_i$; therefore, the groups $J^{ab}{}_i$ and $J^{ac}{}_i$ commute as complexes. Since the groups $J^{ab}{}_i, J^{ac}{}_i, \cdots, J^{ae}{}_i$, where $b, c, \cdots, e$ runs through all the points of $L - \{x\}$, mutually commute, the product $K = J^{ab}{}_iJ^{ac}{}_i\cdots J^{ae}{}_i$ is a group. Each of the groups $J^{ad}{}_i$ has cardinal $n = p^m$, where p is a prime, so the only prime which divides the cardinal of K is p. Thus K is a p-group, and therefore K contains a non-trivial central element β. Since $J^{ab}{}_iJ^{ac}{}_i \supset J^{bc}{}_i$ (as we showed above) we know that K contains all groups of the form $J^{bc}{}_i$, a fortiori, all permutations of the form $\xi^b{}_{ij}\xi^c{}_{ji}$; so β commutes with all these permutations. By Lemma 2.1, G_{xL} is non-trivial. Here x and L were chosen arbitrarily except for the condition $x \in L$, so by Theorem 1.8, $\mathfrak{P}$ is Desarguesian.

3. Fano Planes. We shall call a projective plane a Fano plane if the diagonal points of every quadrangle are collinear.

LEMMA 3.1. *In a Fano plane every permutation of the form* $\xi^a{}_{ij}\xi^b{}_{ji}$ $(a \neq b, i \neq j)$ *has order two. Hence it satisfies* $\xi^a{}_{ij}\xi^b{}_{ji} = \xi^b{}_{ij}\xi^a{}_{ji}$.

Proof. Take any point y on M_i and consider the quadrangle with vertices x, b, $\xi^b{}_{ji}(y)$, and $\xi^a{}_{ij}\xi^b{}_{ji}(y)$. Its diagonal points are a, y, and $\xi^b{}_{ji}\xi^a{}_{ij}\xi^b{}_{ji}(y)$, whence $y = \xi^a{}_{ij}\xi^b{}_{ji}\xi^a{}_{ij}\xi^b{}_{ji}(y)$. Since y is arbitrary $(\xi^a{}_{ij}\xi^b{}_{ji})^2 = \epsilon$.

LEMMA 3.2. *In a Fano plane, the permutations* $\xi^a{}_{ij}\xi^b{}_{ji}$ *and* $\xi^a{}_{ik}\xi^b{}_{ki}$ *commute.*

Proof. In any group, if α, β, and $\alpha\beta$ are of order two, then α and β commute; hence we will prove that $((\xi^a{}_{ij}\xi^b{}_{ji})(\xi^a{}_{ik}\xi^b{}_{ki}))^2 = \epsilon$. Using Lemma 3.1,

$$((\xi^a{}_{ij}\xi^b{}_{ji})(\xi^a{}_{ik}\xi^b{}_{ki}))^2 = (\xi^a{}_{ij}\xi^b{}_{ji})(\xi^b{}_{ik}\xi^a{}_{ki})(\xi^a{}_{ij}\xi^b{}_{ji})(\xi^b{}_{ik}\xi^a{}_{ki})$$
$$= \xi^a{}_{ij}(\xi^b{}_{jk}\xi^a{}_{kj})^2\xi^a{}_{ji} = \xi^a{}_{ij}\xi^a{}_{ji} = \epsilon.$$

LEMMA 3.3. *If a permutation group is abelian and transitive, then it is simply transitive.*

Proof. To prove that there is exactly one element which effects any given partial map $x \to y$, it is sufficient to consider only the case $y = x$ and prove that the only element leaving x fixed is the identity. Suppose α leaves x fixed. For any z we can choose β so that $\beta(z) = x$. Then $\alpha(z) = \beta^{-1}\alpha\beta(z) = z$; so α leaves every point fixed; i.e., α is the identity.

LEMMA 3.4. *In a Fano plane, the sets* $J^{ab}{}_i$ *are elementary abelian 2-groups.*

Proof. We have already seen that $J^{ab}{}_i$ is a set of mutually commuting permutations of $M_i - \{x\}$ which is transitive on $M_i - \{x\}$. Therefore the group generated by $J^{ab}{}_i$ is an abelian transitive permutation group. But such a group is simply transitive, therefore $J^{ab}{}_i$ must be the entire group. Since every element of $J^{ab}{}_i$ (except ϵ) has order two, it is an elementary abelian 2-group.

THEOREM 3.5. *A finite Fano plane is Desarguesian.*

Proof. We know that the sets $J^{ab}{}_i$ are always 2-groups, therefore configuration G holds. If $\mathfrak{P}$ is finite, then the order of $\mathfrak{P}$ is the cardinal of $J^{ab}{}_i$ which is a power of two, so the theorem follows directly from Theorem 2.5.

HARVARD UNIVERSITY.

REFERENCES.

[1] E. Artin, *Selected topics in geometry* (mimeographed), New York University, 1954.

[2] R. Bruck and H. Ryser, "The nonexistence of certain finite projective planes," *Canadian Journal of Mathematics*, vol. 1 (1949), pp. 88-93.

[3] Marshall Hall, Jr., "Projective planes," *Transactions of the American Mathematical Society*, vol. 54 (1943), pp. 229-277.

[4] ———, "Cyclic projective planes," *Duke Mathematical Journal*, vol. 14 (1947), pp. 1079-1090.

[5] W. Klingenberg, "Beweis des Desarguesschen Satzes aus der Reidemeisterfigur und verwandte Sätze, *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, vol. 19 (1955), pp. 158-175.

[6] L. Lombardo-Radici, "Sui piani microdesarguesiani affini," *Atti della Accademia Nazionale Lincei Rendiconti Classe Scienze fisiche Matematiche Naturali*, (8), vol. 15 (1953), pp. 264-271.

[7] H. B. Mann and T. A. Evans, "On simple difference sets," *Sankhya*, vol. 11 (1951), pp. 357-364.

[8] R. Moufang, "Alternativkorper und der Satz von vollständigen Viereck," *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, vol. 9 (1933), pp. 207-222.

[9] G. Pickert, *Projektive Ebene*, Springer, Berlin, 1955.

[10] ———, "Der Satz vom vollständigen Viereck bei kollinearen Diagonalpunkten," *Mathematische Zeitschrift*, vol. 56 (1952), pp. 131-133.

[11] K. Reidemeister, "Topologische Fragen der Differentialgeometrie V," *Mathematische Zeitschrift*, vol. 29 (1929), pp. 427-435.

[12] J. Singer, "A theorem in finite projective geometry and some applications to number theory," *Transactions of the American Mathematical Society*, vol. 43 (1938), pp. 377-385.

[13] M. Zorn, "Theorie der alternativen Ringe," *Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität*, vol. 8 (1931), pp. 123-147.

CONTRIBUTION TO THE PICARD-VESSIOT THEORY OF HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS.*

By A. SEIDENBERG.

1. Introduction. Let F be an ordinary differential field of characteristic 0, C its field of constants, and $L(y) \equiv y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y = 0$, $p_i \in F$, a linear homogeneous differential equation of order n. Let η be a non-trivial solution of this equation and let D be the field of constants of $F\langle\eta\rangle$. One question in the Picard-Vessiot theory is whether η can be so selected that $D = C$. A general result of E. R. Kolchin [3], which is not restricted to linear equations, shows that a solution exists for which D is algebraic over C; and hence disposes of the question if C is algebraically closed. If C is not necessarily algebraically closed, then according to a result of M. P. Epstein [2], one can at any rate choose a fundamental system of solutions $\eta_1, \cdots, \eta_n$ of $L(y)$ so that the field of constants E of $F\langle\eta_1, \cdots, \eta_n\rangle$ is normal over C. The conjecture, however, that one can always arrange to have $D = C$ (hence also $E = C$) turns out to be false, even for $n = 2$, as we show by a counter-example below. Although placed last, this counter-example can be read separately and in itself answers the conjecture. It seems preferable, however, to have a necessary and sufficient condition that $D = C$ be obtainable. This is done below for $n = 2$. This throws light on the counter-example and also prepares the way, possibly, for consideration of the case $n > 2$.

2. Reduction to a first order equation. For a first order equation $y' = py$ one can obtain $D = C$ without difficulty and a proof can be found in [2]. One may show, in fact, (by an argument occurring in Section 3, below) that if the general solution[1] y of $y' = py$ introduces a constant, then

* Received July 5, 1956.

[1] By a *solution* to a differential equation $G(Y, Y_1, \cdots, Y_r) = 0$, where $G \in F\{Y\} = F[Y, Y_1, \cdots]$ we mean a quantity y in a differential extension field of the base field F such that $G(y, y', \cdots, y^{(r)}) = 0$. If $G(Y)$ is an irreducible polynomial involving Y_r, then there is a solution y such that degree of transcendence of $F\langle y\rangle/F = r$; and $F\langle y\rangle/F$ is uniquely determined up to an isomorphism by this condition: we refer to this solution as the *general solution* of $G(Y) = 0$. Caution: it is not in general true that every solution of $G(Y) = 0$ is a specialization of the general solution: for illustrative examples, see J. F. Ritt's Colloquium Series book, *Differential Algebra*,

there is a constant of the form $f = \sum a_i y^i$, $a_i \in F$, $f \notin F$, that is, one that is a polynomial in the algebraic indeterminate y. Thus $a_i' + ia_i p = 0$ for $i = 0, 1, \cdots$; and for at least one $i > 0$, $a_i' + ia_i p = 0$, $a_i \neq 0$. If r is the least positive integer for which there is an $a \in F$ such that $a' + rap = 0$, $a \neq 0$, then $a^{-1/r}$ is a solution to $y' = py$ which introduces no new constant.

Let

$$y'' = qy' + py \tag{1}$$

be the given differential equation. If $p = 0$, then we can solve $y'' = qy'$ for y' in such manner that $F\langle y'\rangle$ has C as field of constants. Let b be such a solution, $F_1 = F\langle b\rangle$. By arguments similar to those of the last paragraph, one can also solve $y' = b$ without introducing new constants (see [2]). We might therefore assume, if convenient, that $p \neq 0$; actually we will make no such restriction.

Place $y'/y = u$. Then $y''/y = (y'/y)' + (y'/y)^2$, so that equation (1) can be writen equivalently as

$$u' = p + qu - u^2. \tag{2}$$

This equation will have a solution introducing no new constant if and only if (1) has a (non-trivial) solution introducing no new constant.

3. Constants obtained from the general solution of (2). Assume that the general solution u of (2) introduces a new constant, and let one such be $f(u)/g(u)$, $f(u)$, $g(u) \in F[u]$, $(f(u), g(u)) = (1)$. Note that no polynomial $h(u) \in F[u]$, $h \notin F$, can be constant, for $\deg h'(u) = 1 + \deg h(u)$, where "deg" abbreviates "degree"; here $h'(u)$ is the polynomial obtained by differentiating $h(u)$ and replacing u' by $p + qu - u^2$. We say that $\deg f = \deg g$. In fact, suppose $\deg f > \deg g$. Then

$$f/g = P + f_1/g, \qquad \deg f_1 < \deg g, \qquad f_1 \neq 0, P \neq 0, \deg P > 0;$$
$$0 = P' + (f_1'g - f_1g')/g^2.$$

This is a contradiction, since $(f_1'g - f_1g')/g^2$, even if a polynomial, is at most of degree 0, whereas $\deg P' > \deg P > 0$. Hence:

LEMMA 1. *If $f(u)/g(u)$ is a constant $(\neq 0)$, then $\deg f = \deg g$.*

The same result, with the same proof, holds if instead of (2) we have the equation

$$u' = p + qu + ru^2, \quad r \neq 0. \tag{3}$$

---

pp. 24, 25, 32, 33. In the case of linear equations, however, and more generally in the case that $\partial G/\partial Y_r$ is a constant, a condition that obtains below, it is true that every solution is a specialization of the general solution.

Placing $v = 1/u$, (3) becomes

$$v' = -r - qv - pv^2, \tag{4}$$

that is, an equation of the same form as (3) if, as we asume for the present, $p \neq 0$. Writing $f(u)/u^{\deg f} = f_1(v)$, $g(u)/u^{\deg g} = g_1(v)$, we have $f(u)/g(u) = f_1(v)/g_1(v)$ and $\deg f_1(v) = \deg g_1(v)$; here we make use of the assumption $p \neq 0$. Hence:

LEMMA 2. *Assuming $p \neq 0$, if $f(u)/g(u)$ is a constant $(\neq 0)$ and $(f, g) = (1)$, then $f(0) \neq 0$ and $g(0) \neq 0$.*

THEOREM 1. *The field of constants C' of $F\langle u\rangle$, where u is the general solution of (2), is either C or is a field of algebraic functions of one variable over C of genus 0. In the latter case, a necessary condition that (2) have a solution which preserves constants is that C' be a simple transcendental extension of C.*

Proof. As in the lemmas, we assume $C' \supset C$ properly, and will prove C' is a field of algebraic functions of one variable over C of genus 0. It is known ([5; Cor. 3, p. 767]) that $\text{d.t.}\, F(C')/F = \text{d.t.}\, C'/C$, hence $\text{d.t.}\, C'/C = 1$; here "d.t." abbreviates "degree of transcendence." Moreover it is not difficult to see that C'/C is finite. Let, then, $x, y \in C'$ which generate C'/C: $C' = C(x, y)$. Let x, y satisfy the irreducible polynomial $G(X, Y) \in C[X, Y]$. Since C is algebraically closed in F, the polynomial $G(X, Y)$ is still irreducible in $F[X, Y]$ ([6; p. 371]). Since not both x, y are algebraic over F, $G(X, Y)$ is the irreducible polynomial satisfied by x, y over F. By [1; Th. 5, p. 99], genus $F(x, y)/F = \text{genus}\, C(x, y)/C$. Since $F(x, y) \subseteq F(u)$, this genus is 0.

Let, now, θ be a solution of (2) introducing no new constants. The assignment $u \to \theta$ determines a place of $F(u)/F$ and this place induces a place in $C(x, y)/C$. If $f(u)/g(u) \in C(x, y)$, $(f, g) = (1)$, and the residue of f/g is finite, then $g(\theta) \neq 0$ and this residue is $f(\theta)/g(\theta)$. Since, clearly, $f(\theta)/g(\theta)$ is a constant, $f(\theta)/g(\theta) \in C$. Thus the place induced in $C(x, y)/C$ is of degree 1, and this implies that $C(x, y)$ is a simple transcendental extension of C ([1; Section 2, p. 23]).

As already remarked, the necessary condition of Theorem 1 does not always obtain: but we postpone giving a counter-example to the end. Instead we proceed now to show that the condition is also sufficient.

4. The case F finite over C. We continue to assume that the general solution of (2) introduces a constant; and let $f(u)/g(u)$ be as before. If

a solution θ to (2) exists such that the constant field of $F\langle\theta\rangle$ is C, then clearly $f(\theta)/g(\theta) = c \in C$, at least if $g(\theta) \neq 0$. Thus θ is a solution of the equation $f(u) - cg(u) = 0$ for some $c \in C$, or of $g(u) = 0$. Conversely we have:

THEOREM 2. *If $f(u)/g(u)$, $(f, g) = (1)$, is a constant $(\neq 0)$, then any solution of $f(u) = 0$, is also a solution of (2).*

Proof. Let θ be a solution of $f(u) = 0$. Placing $u = \theta + v$, equation (2) becomes

$$v' + \theta' = p + q\theta - \theta^2 + (q - 2\theta)v - v^2, \tag{5}$$

while we can write $f(u)/g(u) = f_1(v)/g_1(v)$, $f_1, g_1 \in F\langle\theta\rangle[v]$. Here $f_1(0) = 0$, $g_1(0) \neq 0$. By Lemma 2, this is a contradiction unless $\theta' = p + q\theta - \theta^2$.

In proceeding, we will apply a well-known theorem of Bertini on linear systems on an algebraic variety (see [9]). It will be convenient, however, to formulate as much of this theorem as we will need in invariantive, or field-theoretic, form. For the next theorem, then, and only for it, the fields F and C will be fields (of characteristic 0) in the algebraic (or abstract) sense.

THEOREM 3. *Let C and F be fields, C algebraically closed in F, and F a finite extension of C. Let u be an indeterminate, $f(u), g(u) \in F[u]$, $(f, g) = (1)$, and not both f, g in F; and assume that $C(f(u)/g(u))$ is algebraically closed in $F(u)$. Then for all but a finite number of $c \in C$, $f(u) - cg(u)$ is irreducible in $F[u]$ and C is algebraically closed in $F(\theta)$, where θ is a root of $f(u) - cg(u) = 0$.*

Proof. Let d.t. $F/C = r - 1$. It is known that there exist generators $\xi_1, \cdots, \xi_n$ of F/C such that $C[t, t\xi_1, \cdots, t\xi_n]$ is integrally closed, where t is an indeterminate over $F = C(\xi_1, \cdots, \xi_n)$; see [8; Section 20, p. 290 and Definition, Section 18, p. 285] and [10; p. 508]. By [8; p. 285], $\xi_1, \cdots, \xi_n$ are the coördinates of the general point of a normal model of F/C (see [9; Footnote 2, p. 49]). Then, as one easily verifies, $\xi_1, \cdots, \xi_n, u$ is the general point of an r-dimensional normal variety V_r. The field $C(f(u)/g(u))$ defines a non-composite pencil of V_{r-1}'s on V_r (see [9]). Multiplying f and g by an appropriate element of F, we assume without loss of generality that $f, g \in C[\xi_1, \cdots, \xi_n, u]$. Let P be an $(r-1)$-dimensional prime ideal of $(f(u) - cg(u))$. It is possible that $g(u) \in P$, whence also $f(u) \in P$, and P will be a prime ideal of an element $h(\xi) \in C[\xi_1, \cdots, \xi_n]$; here P will not

depend on c. Except for this possibility, the variety of P corresponds to the place $f(u)/g(u) \to c$. By Bertini's Theorem [9; Sections 10, 12, pp. 61, 64], then, for all but a finite number of $c \in C$, at most one $(r-1)$-dimensional prime ideal P contains $f(u) - cg(u)$ but not $g(u)$; also the value of $f(u)/g(u) - c$ in the valuation determined by P is 1 (the integers having been taken as value group), and likewise the value of $f(u) - cg(u)$ is 1; and C is algebraically closed in the quotient field of $C[\xi_1, \cdots, \xi_n, u]/P$. Using the well-known fact that a principal ideal $(\neq(0), \neq(1))$ in an integrally closed finite integral domain is unmixed $(r-1)$-dimensional, we have that the normal decomposition of $(f(u) - cg(u))$ is of the form $P \cap Q_1 \cap \cdots \cap Q_s$ (or of the form $Q_1 \cap \cdots \cap Q_s$), where the Q_i are primary ideals belonging to $(r-1)$-dimensional prime ideals P_i containing an element $h(\xi) \in C[\xi_1, \cdots, \xi_n]$, $h \neq 0$: here P, that is, a prime ideal not containing any $h(\xi)$, $h \neq 0$, must occur, since

$$C(\xi_1, \cdots, \xi_n)[u] \cdot (f(u) - cg(u)) \neq (1).$$

Thus

$$C(\xi_1, \cdots, \xi_n)[u] \cdot (f(u) - cg(u)) = C(\xi_1, \cdots, \xi_n)[u] \cdot P$$

is prime and C is algebraically closed in $F[u]/(f(u) - cg(u))$.

To apply this theorem, we let F, C be as in the foregoing with the additional assumption that F/C is finite, in the differential sense: $F = C\langle v_1, \cdots, v_s \rangle$. This still does not make F/C finite in the algebraic, or abstract, sense. For a moment, then, assume that F/C is also (differentially) algebraic, so that F/C is also finite in the algebraic sense. With a view towards a sufficiency condition, let $t = f(u)/g(u)$ be such that $C(t)$ is the field of constants of $F(u)$. Then one applies immediately Theorem 3 (and Theorem 2) to get a solution θ of (2) such that C is the field of constants of $F\langle\theta\rangle$. Here θ can be the solution of any equation $f(u) - cg(u) = 0$, with at most a finite number of exceptions on c (of course, such solution exists; see, for example, [7; p. 179, Lemma]).

To remove the finiteness condition on F/C (that is, the condition that F/C be algebraic), let d.d.t. $F/C = r$, $r \leq s$. Let

$$F_{r+i}(v_1, \cdots, v_r; v_{r+1}, \cdots, v_{r+i-1}; v_{r+i}) = 0$$

be the irreducible polynomial relation defining v_{r+i} as differentially algebraic over $C\langle v_1, \cdots, v_{r+i-1} \rangle$, $i = 1, \cdots, s-r$. Considering F/C in the abstract sense, it is not difficult to see that F *is a pure transcendental extension of a*

finite extension F_1 of C. In fact, let d_{s-1} be the order of F_{s-1} in v_{s-1}, and let M be a bound on the j of the v_{s-1j} actually occurring in F_s. If $M - d_{s-1} = k_1 > 0$, then to the set S of the relations $F_{r+1} = 0, \cdots, F_{s-1} = 0$, $F_s = 0$ we adjoin the relations $F_{s-1}' = 0, \cdots, F_{s-1}^{(k_1)} = 0$ to obtain the set

$$S_1 = \{F_1 = 0, \cdots, F_{s-2} = 0; F_{s-1} = 0, \cdots, F_{s-1}^{(k_1)} = 0; F_s = 0\}.$$

Similarly, if d_{s-2} is the order of F_{s-2} in v_{s-2}, and M_1 is a bound on the j of the v_{s-2j} actually occurring in $F_{s-1}, \cdots, F_{s-1}^{(k_1)}, F_s$, and if $M_1 - d_{s-2} = k_2 > 0$, then to S_1 we adjoin the relations $F_{s-2}' = 0, \cdots, F_{s-2}^{(k_2)} = 0$, to obtain the set S_2. And so we proceed until we have a set

$$T = \{F_{r+1} = 0, \cdots, F_{r+1}^{(k_s-r-1)} = 0; \cdots; F_{s-1}^{(k_1)} = 0; F_s = 0\}.$$

The set T is such that the maximum of j for the variables v_{r+ij} actually occurring in T is already reached by the v_{r+ij} occurring in the relations preceding $F_{r+i+1} = 0$ $(i > 0)$. Let F_1 be the field obtained from C by adjoining all the v_{ij} actually occurring in the relations of T. Then F_1 is a finite extensions of C of the type sought. One sees this easily enough if one first adjoins the appropriate v_{ij}, $i \leqq r$; then the appropriate v_{r+1j} as successively determined from $F_{r+1} = 0, \cdots, F_{r+1}^{(k_s-r-1)} = 0$; then the v_{r+2j}, etc. The v_{ij}, $i \leqq r$, not thus adjoined clearly form a pure transcendancy basis of F/F_1. Moreover *we may suppose* F_1 *to contain any finite number of preassigned* v_{ij}, in particular those occurring in $f(u)$, $g(u)$, p, and q. In fact, some of these may already be in the F_1 constructed; and the others are rationally dependent over F_1 on a finite number of the v_{ij}, $i \leqq r$. Adjoining these v_{ij}, $i \leqq r$, to F_1 we get an F_1 of the desired kind. Now, by Theorem 3, C is algebraically closed in $F_1(\theta)$, where θ is a root of $f(u) - cg(u) = 0$ for appropriate $c \in C$. Adjoining the pure transcendancy basis, C remains algebraically closed in $F(\theta)$. Finally, as remarked at the end of the last paragraph, we regard $F(\theta) = F\langle\theta\rangle$ as a differential field extension of F. *This completes the proof of the main theorem* (*Theorem* 4, *below*) *in the case that* F/C *is finite.*

5. The main theorem. Let F_0 be a finite extension of C containing p, q and the coefficients of $f(u)$, $g(u)$, where $C(f(u)/g(u))$ is the field of constants of $F(u)$; $f, g \in F[u]$, $(f, g) = (1)$. By Section 4, there exists a (non-trivial) solution η_1 of $y'' = qy' + py$ such that C is the field of constants of $F_0\langle\eta_1\rangle$. Placing $\eta_2 = \eta_1 z$, we find

$$(2\eta_1' - q\eta_1)z' + \eta_1 z'' = 0 \tag{6}$$

as the condition on z in order that η_2 satisfy (1). By what has been said in the first two paragraphs of Section 2, one can find another solution η_2 of (1), such that C is the constant-field of $F_0\langle\eta_1, \eta_2\rangle$ and η_2/η_1 is not constant, that is, $z' \neq 0$. The solutions η_1, η_2 form a so-called *fundamental system of solutions* of (1).

The arguments of the next two lemmas are largely familiar from the papers of E. R. Kolchin; see in particular [4; Section 17, p. 29] and [5; Section 5, p. 768].

Let C_1 be the field of constants of $F\langle\eta_1, \eta_2\rangle$. Then C_1 is a finite extension of C. We wish to consider now the isomorphisms over F of $F\langle\eta_1, \eta_2\rangle$ into $F\langle\eta_1, \eta_2\rangle(\bar{C}_1)$, where $\bar{C}_1$ is the algebraic closure of C_1. An isomorphism over F of $F\langle\eta_1, \eta_2\rangle$ into a universal domain Ω over $F\langle\eta_1, \eta_2\rangle(\bar{C}_1)$ will send η_i into $\bar{\eta}_i = c_{i1}\eta_1 + c_{i2}\eta_2$, where c_{i1}, c_{i2} are constants. Since also $\bar{\eta}_1' = c_{i1}\eta_1' + c_{i2}\eta_2'$, one has that $c_{ij} \in F\langle\eta_1, \eta_2, \bar{\eta}_1, \bar{\eta}_2\rangle$; therefore the c_{ij} are constants in $F\langle\eta_1, \eta_2\rangle(\bar{C}_1)$, hence are in $\bar{C}_1$ (see [5; Cor. 5, p. 768]). Conversely, an isomorphism with the $c_{ij} \in \bar{C}_1$ will be into $F\langle\eta_1, \eta_2\rangle(\bar{C}_1)$.

The elements η_1, η_1', η_2, η_2' determine an ideal in the polynomial ring $F[Y_1, Y_1', Y_2, Y_2']$. Let $G_1, \cdots, G_s$ be a basis for this ideal. If $\eta_i \to \bar{\eta}_i = c_{i1}\eta_1 + c_{i2}\eta_2$ is to be an isomorphism over F into the universal domain Ω, then $G_i(\bar{\eta}_1, \bar{\eta}_1', \bar{\eta}_2, \bar{\eta}_2') = G_i(c_{i1}\eta_1 + c_{i2}\eta_2, \cdots) = 0$: these are obviously necesary conditions on the c_{ij}. Conversely if the c_{ij} satisfy these conditions, then $\eta_i \to \bar{\eta}_i$ clearly determines (by substitutions) at least a (differential) *homomorphism* over F into Ω. If in addition the c_{ij} are in $\bar{C}_1$ and $\begin{vmatrix} c_{11}c_{12} \\ c_{21}c_{22} \end{vmatrix} \neq 0$, then the homomorphism is an isomorphism: in fact, following an observation of Kolchin [4; Section 17, p. 30] in a similar situation, we note that $F\langle\eta_1, \eta_2\rangle(\bar{C}_1) = F\langle\bar{\eta}_1, \bar{\eta}_2\rangle(\bar{C}_1)$, so

$$\text{d. t. } F(\eta_1, \eta_1', \eta_2, \eta_2')/F = \text{d. t. } F(\bar{\eta}_1, \bar{\eta}_1', \bar{\eta}_2, \bar{\eta}_2')/F,$$

so the homomorphism must be an isomorphism. *Thus we have obvious necessary and sufficient conditions on the c_{ij} in order that $\eta_i \to \bar{\eta}_i = c_{i1}\eta_1 + c_{i2}\eta_2$ define an isomorphism of $F\langle\eta_1, \eta_2\rangle$ into $F\langle\eta_1, \eta_2\rangle(\bar{C}_1)$.* Hence we have the following lemma.

LEMMA 3. *Let η_1, η_2 be a fundamental system of solutions over F for the equation $y'' = qy' + py$; and let $G_1, \cdots, G_s \in F[Y_1, Y_1', Y_2, Y_2']$ be a basis for the ideal of polynomials satisfied by $\eta_1, \eta_1', \eta_2, \eta_2'$. Let F_1 be a subfield of F containing p, q and the coefficients of the G_i; C_1, the constant-field of $F\langle\eta_1, \eta_2\rangle$, D_1, the constant-field of $F_1\langle\eta_1, \eta_2\rangle$. Then any isomorphism over*

F_1 of $F_1\langle\eta_1,\eta_2\rangle$ into $F_1\langle\eta_1,\eta_2\rangle(\bar{D}_1)$ can be extended, in one and only one way, to an isomorphism over F of $F\langle\eta_1,\eta_2\rangle$ into $F\langle\eta_1,\eta_2\rangle(\bar{C}_1)$.

By a similar argument we obtain the following.

LEMMA 4. *Let G be a differential field between F and $F\langle\eta_1,\eta_2\rangle$. Then any isomorphism over F of G into $F\langle\eta_1,\eta_2\rangle(\bar{C}_1)$ can be extended (in at least one way) to an isomorphism of $F\langle\eta_1,\eta_2\rangle$ into $F\langle\eta_1,\eta_2\rangle(\bar{C}_1)$.*

Proof. The given isomorphism σ can at least be extended to an isomorphism τ of $F\langle\eta_1,\eta_2\rangle$ into the universal domain Ω (see [5; Section 5, p. 768]). Let τ be given by $\eta_i \to \bar{\eta}_i = c_{i1}\eta_1 + c_{i2}\eta_2$.

The field G is generated by a finite number of elements $r_i(\eta)/s_i(\eta)$, $r_i(\eta)$, $s_i(\eta) \in F[\eta_1,\eta_1',\eta_2,\eta_2']$. Let their images under σ be $\bar{r}_i(\eta)/\bar{s}_i(\eta), \bar{r}_i$, $\bar{s}_i \in F(\bar{C}_1)[\eta_1,\eta_1',\eta_2,\eta_2']$. Then $r_i(\bar{\eta})/s_i(\bar{\eta}) = \bar{r}_i(\eta)/\bar{s}_i(\eta)$. Consider the following finite set S of polynomial equations:

$$r_i(C_{11}\eta_1 + C_{12}\eta_2, \cdots)\bar{s}_i(\eta) = s_i(C_{11}\eta_1 + C_{12}\eta_2, \cdots)\bar{r}_i(\eta), \quad i = 1, 2, \cdots;$$
$$G_j(C_{11}\eta_1 + C_{12}\eta_2, \cdots) = 0.$$

By [5; Cor. 3, p. 767] there exists an equivalent finite set T of polynomial equations over $\bar{C}_1$. Then the set T, together with the inequality $C_{11}C_{22} - C_{12}C_{21} \neq 0$, has a solution, namely, the c_{ij} form a solution. Hence, by Hilbert's Nullstellensatz, T also has a solution $\bar{c}_{ij}$ with the $\bar{c}_{ij} \in \bar{C}_1$ and $\bar{c}_{11}\bar{c}_{21} - \bar{c}_{12}\bar{c}_{21} \neq 0$. Then $\eta_i \to \bar{c}_{i1}\eta_1 + \bar{c}_{i2}\eta_2$ is an isomorphism of the required kind.

THEOREM 4. *Let F be a differential field with field of constants C, $y'' = qy' + py$ a differential equation with $q, p \in F$; and assume that the field of constants of $F\langle u\rangle$ is a simple transcendental extension of C, where $u' = p + qu - u^2$. Then the equation $y'' = qy' + py$ has a solution $\eta_1 \neq 0$ such that the field of constants of $F\langle\eta_1\rangle$ is still C. The solution η_1 can be completed to a fundamental system of solutions η_1, η_2 so that the field of constants of $F\langle\eta_1,\eta_2\rangle$ is still C.*

Proof. Let F_0, η_1, η_2 be taken as in the first paragraph of this section; η_1, η_2 will not necessarily be our eventual η_1, η_2. Let C_1 be the field of constants of $F\langle\eta_1,\eta_2\rangle$; and let F_1 be a finite extension of F_0 such that $F_1\langle\eta_1,\eta_2\rangle$ contains C_1, and the coefficients of $G_1, \cdots, G_s$, a basis for the polynomials over F satisfied by $\eta_1,\eta_1',\eta_2,\eta_2'$; F_1 a subfield of F. Let $\theta_d = (\eta_1' + d\eta_2')/(\eta_1 + d\eta_2)$, $d \in C$. Then θ_d satisfies the equation (2). The

element θ_d takes on infinitely many values as d varies over C; and we will suppose $g(\theta_d) \neq 0$, where $f(u)/g(u)$ is the constant considered previously ($C(f/g)$ is the constant-field of $F\langle u\rangle$). Since $f(\theta_d)/g(\theta_d)$ is constant and $\theta_d \in F_0\langle\eta_1, \eta_2\rangle$, we have $f(\theta_d)/g(\theta_d) = c(d) \in C$ for all d (for which $g(\theta_d) \neq 0$). Clearly $c(d)$ takes on infinitely many values as d varies over C. Hence we can certainly take d such that the field of constants of $F_1(\theta_d)$ be still C. Then the field of constants of $F(\theta_d)$ will also be C. In fact, let $\phi \in C_1$, $\phi \notin C$. Then there exists an isomorphism σ of $F_1(\theta_d)(\phi)$ into $F_1\langle\eta_1, \eta_2\rangle(\bar{C}_1)$ which is the identity on $F_1(\theta_d)$ but moves ϕ: namely, if ϕ is algebraic over C we construct such an isomorphism by mapping ϕ into one of its conjugates over $F_1(\theta_d)$ (= its conjugates over C), while if ϕ is not algebraic over C, then we may, similarly, map ϕ into, say, $\phi + 1$. Thus we are assured of the existence of σ. By Lemma 4, σ can be extended to an isomorphism of $F_1\langle\eta_1, \eta_2\rangle$ into $F_1\langle\eta_1, \eta_2\rangle(\bar{C}_1)$ and then, by Lemma 3, to an isomorphism τ over F of $F\langle\eta_1, \eta_2\rangle$ into $F\langle\eta_1, \eta_2\rangle(\bar{C}_1)$. We have $\tau(\theta_d) = \theta_d$, $\tau(\phi) \neq \phi$, hence $\phi \notin F(\theta_d)$. Since this holds for every $\phi \in C_1$, $\phi \notin C$, we conclude that the field of constants of $F(\theta_d)$ is C. This completes the proof.

6. The counter-example. Let R = field of real numbers with trivial differentiation. To R adjoin the general solution a of

$$4a^2 + a'^2 = -1. \tag{1}$$

Place $F = R\langle a\rangle = R(a, a')$. Let C = field of constants of $R\langle a\rangle$. Were C not algebaric over R, then we would have $R(a, a')$ algebraic over C, hence $R(a, a') = C$, which is not so since $a' \neq 0$. So C is algebraic over R. Since $4a^2 + a'^2 + 1 = 0$ is absolutely irreducible, we have $C = R$ ([6; p. 371]).

From (1) we obtain $8aa' + 2a'a'' = 0$, whence

$$a'' = -4a. \tag{2}$$

Let $L(y) \equiv y'' + y$ and consider the equation

$$y'' + y = 0. \tag{3}$$

Placing $y'/y = u$, we see that any non-trivial solution η of (3) yields a solution η'/η to

$$u' = -1 - u^2. \tag{4}$$

We will show that *if θ is any solution of $u' = -1 - u^2$, then $F\langle\theta\rangle$ contains a constant not in R.*

Let u be the general solution of $u' = -1 - u^2$. One finds:

$$(1+u^2)' = 2uu' = -2u(1+u^2)$$
$$(a + a'u - au^2)' = (a'' + 2a)u - 2a'u^2 + 2au^3$$
$$= -2u(a + a'u - au^2) \quad \text{(by (2))},$$

whence $(a + a'u - au^2)/(1+u^2)$ is constant. Thus the general solution u introduces new constants.

Let θ be any solution of $u' = -1 - u^2$. If $1 + \theta^2 = 0$, then $F\langle\theta\rangle$ contains i, $i = \sqrt{-1} \notin R$. We proceed, assuming $1 + \theta^2 \neq 0$. Then $(a + a'\theta - a\theta^2)/(1+\theta^2)$ is a constant. If it is not in R, then it is a new constant; and if $(a + a'\theta - a\theta^2)/(1+\theta^2) = c \in R$, then

$$(a - c) + a'\theta - (a + c)\theta^2 = 0,$$

from which one finds that

$$[a'^2 + 4(a^2 - c^2)]^{\frac{1}{2}} \in F\langle\theta\rangle,$$

that is, by (1), that

$$(-1 - 4c^2)^{\frac{1}{2}} \in F\langle\theta\rangle.$$

Hence also $i \in F\langle\theta\rangle$.

Remark. The constant-field of $F\langle u\rangle$, in the present case, is $R(x, y)$, where $x = (a + a'u - au^2)/(1+u^2)$, $y = (a' - 4au - a'u^2)/(1+u^2)$. In fact, one verifies directly that x and y are constants. Moreover, by Lemma 2, any constant k $(\neq 0)$ is of the form $f(u)/g(u)$, $f(u)$, $g(u) \in F[u]$, $f(0) \neq 0$, $g(0) \neq 0$. Let

$$f(u) = c_0(a, a') + c_1(a, a')u + \cdots,$$
$$g(u) = d_0(a, a') + d_1(a, a')u + \cdots,$$

where we may suppose the $c_i, d_j \in R[a, a']$. Then $f(u)/g(u) - c_0(x, y)/d_0(x, y)$ evaluated at $u = 0$ is 0 and therefore $f(u)/g(u) - c_0(x, y)/d_0(x, y) = 0$, that is, $k = c_0(x, y)/d_0(x, y) \in R(x, y)$. Note also, by an argument of the type just made, that the constants x, y satisfy the relation $4x^2 + y^2 = -1$.

UNIVERSITY OF CALIFORNIA,
BERKELEY 4, CALIFORNIA.

REFERENCES.

[1] C. Chevalley, "Introduction to the theory of algebraic functions of one variable" (Mathematical Surveys VI, American Mathematical Society), New York, 1951.

[2] M. P. Epstein, "An existence theorem in the algebraic study of homogeneous linear ordinary differential equations," *Proceedings of the American Mathematical Society*, vol. 6 (1955), pp. 33-41.

[3] E. R. Kolchin, "Existence theorems connected with the Picard-Vessiot theory of homogeneous linear ordinary differential equations," *Bulletin of the American Mathematical Society*, vol. 54 (1948), pp. 927-932.

[4] ———, "Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations," *Annals of Mathematics*, vol. 49 (1948), pp. 1-42.

[5] ———, "Galois theory of differential fields," *American Journal of Mathematics*, vol. 75 (1953), pp. 753-824.

[6] A. Seidenberg, "The hyperplane sections of normal varieties," *Transactions of the American Mathematical Society*, vol. 69 (1950), pp. 357-386.

[7] ———, "Some basic theorems in differential algebra (characteristic p, arbitrary), *Transactions of the American Mathematical Society*, vol. 72 (1952), pp. 174-190.

[8] O. Zariski, "Some results in the arithmetic theory of algebraic varieties," *American Journal of Mathematics*, vol. 61 (1939), pp. 249-294.

[9] ———, "Pencils on an algebraic variety and a new proof of a theorem of Bertini," *Transactions of the American Mathematical Society*, vol. 50 (1941), pp. 48-70.

[10] ———, "Foundations of a general theory of birational correspondences," *Transactions of the American Mathematical Society*, vol. 53 (1943), pp. 490-542.

CAUCHY'S STABLE DISTRIBUTIONS AND AN "EXPLICIT FORMULA" OF MELLIN.*

By AUREL WINTNER.

---

1. If α (or its real part) is positive and less than 1, and if t (or its real part) is positive, then, according to Mellin, the case $\phi(x) = \exp(-tx^\alpha)$ of the Euler-MacLaurin difference $\Sigma\phi(n) - \int\phi(x)dx$, where $n = 1, 2, \cdots$ and $0 \leqq x < \infty$, can be evaluated explicitly; cf. (9) and (9 bis) below. The resulting "explicit formula" turns out to be closely connected with results which in earlier papers I obtained on Cauchy's symmetric stable distribution functions of index α (where either $0 < \alpha < 1$, as in Mellin's formula, or, to certain ends, $0 < \alpha < 2$, as in Paul Lévy's justification of Cauchy's formal result). This family of distribution functions is defined by the property that the Fourier transform of the distribution is even and is identical with e^{-t^α} if $0 \leqq t < \infty$.

In what follows, the analytical questions concerning Cauchy's distributions will be dealt with in a systematic way. Roughly speaking, there will be three issues involved: (i) the explicit form of the Laplace transform of the distribution, (ii) the connection of the Laplace transform with the Stieltjes transform, finally (iii) the application of the resulting explicit formulae to a determination of the weight function which belongs to the "stratification," in terms of the symmetric Gaussian distribution ($\beta = 2$), of the symmetric stable distribution of any index $\beta < 2$.

At the end (Section 13), there will be considered the asymptotic expansion (and its implications for Cauchy's transcendents) which takes the place of Mellin's convergent expansion if $0 < \alpha < 1$ is replaced by $1 < \alpha < \infty$ (the limiting case $\alpha = 1$ is trivial). The values $\alpha = 2n$, where $n = 2, 3, 4, \cdots$, are exceptional (but $n = 1$, i.e., $\alpha = 2$, is trivial). For this exceptional case, there will be reproduced with Professor Pólya's permission an asymptotic formula which he communicated to me several years ago.

For $0 < \alpha < 2$, a side issue will be the "shape" of the frequency curves. This issue has, however, little to do with Cauchy's particular choice of symmetric distributions, since what results holds for all symmetric distributions

---

* Received June 11, 1956.

of the so-called L-class (Khintchine-Lévy). This and related results are dealt with in Appendix II and Appendix III.

Appendix I deals again with the entire range $0 < a < \infty$ but in the angular case. What is then involved is the extension from the case $a = 2$ of an elliptic theta to the case of an arbitrary $a > 0$ (including the exceptional a-values $4, 6, \cdots$) of the function

$$1 + 2 \sum_{n=1}^{\infty} q^{n^a} \cos nx.$$

This function has of course often been considered. The results of Appendix I are however substantially sharper than those obtained in the literature consulted.

2. Replace the function e^{-t^a}, where $0 \leqq t < \infty$, by its Fourier cosine transform

$$F(t) = \int_0^{\infty} e^{-s^a} \cos ts \, ds. \tag{1}$$

Then a partial integration shows that

$$F(t) = G(1/t^a)/t^{1+a}, \tag{2}$$

where

$$G(t) = \int_0^{\infty} e^{-ts} \sin(s^{1/a}) \, ds \tag{3}$$

(even if only just $0 < a < \infty$, rather than $0 < a < 1$, is assumed).

The representation (2) of (1) in terms of the Laplace transform (3) was pointed out in [13], p. 678. The correspondence (2) between F and G is invariant under the substitution

$$(F, t, a) \to (G, 1/t, 1/a) \tag{2*}$$

(if $0 < t < \infty$ and $0 < a < \infty$), since (2) is equivalent to

$$G(t) = F(1/t^{1/a})/t^{1+1/a}. \tag{2 bis}$$

The alternative devices, $F \to G$ and $G \to F$, can be applied to each of the Fourier analyses to be considered.

An immediate consequence of (2) and (3) is the following asymptotic relation of Pólya (cf. Pólya-Szegö, chap. III, no. 154):

$$F(t) \sim At^{-1-a} \text{ as } t \to \infty, \text{ where } A = \Gamma(1 + a) \sin(\tfrac{1}{2}\pi a) \tag{4}$$

(and where the $\sim$ is meant in the sense that

(4 bis) $$F(t) = o(t^{-1-\alpha}) \text{ as } t \to \infty \text{ if } \alpha = 2, 4, \cdots,$$

the constant A being 0 when α is an even integer). In fact, it is well-known that

$$\int_0^\infty \sin(s^{1/\alpha})\, ds = \Gamma(1+\alpha) \sin(\tfrac{1}{2}\pi\alpha),$$

provided that integral on the right, which is convergent only if $0 < \alpha < 1$, is meant as a Cesàro limit (of an appropriate positive index) when $1 \leqq \alpha < \infty$. Since summability in Abel's sense is implied by (and is a process consistent with) summability in Cesàro's sense, (4) follows by using (2) for $t \to \infty$ and letting $t \to +0$ (Abel) in (3).

It is clear from (1) that $F(t) > 0$ holds for $t = 0$, hence for all sufficiently small $t > 0$, while (4) shows that, if $2n < \alpha < 2(n+1)$, where $n = 0, 1, \cdots$, then $F(t) > 0$ or $F(t) < 0$ holds for all sufficiently large t according as n is even or odd. (In view of (4 bis), nothing follows for large t if $\alpha = 2, 4, \cdots$, but Pólya has shown that (1) is then an entire function having no zero at all or only real zeros according as $\alpha = 2$ or $\alpha = 4, 6, \cdots$; cf. Pólya-Szegö, chap. V, no. 170).

3. Let $\psi(s)$, where $0 \leqq s < \infty$, be a real-valued (not necessarily continuous) function whose Laplace transform

(5) $$\int_0^\infty e^{-ts} \psi(s)\, ds$$

is convergent for $t > 0$. Let N, where $0 \leqq N \leqq \infty$, denote the number of the positive zeros t of (5), and let M, where $0 \leqq M \leqq \infty$, denote the number of those s-values at which the function

(5 bis) $$\Psi(s) = \int_0^s \psi(u)\, du$$

changes sign when s increases from $s = +0$ to ∞. Then $N \leqq M$. This inequality expresses a theorem of Laguerre as further developed by Pólya (cf. Pólya-Szegö, chap. V, no. 82).

Due to (2) and (3), the estimate $N \leqq M$ can be applied to (1), with

(6) $$\psi(s) = \int_0^s \sin(u^{1/\alpha})\, du$$

(so that

$$\Psi(s) = \int_0^s (s-u)\sin(u^{1/a})\,du, \tag{6 bis}$$

by (5 bis)). In fact, if $N = N_a$ and $M = M_a$ denote the respective number of times the functions (1) and (6 bis) change sign when t and s vary from $+0$ to ∞, then it follows that

$$0 \leqq N_a \leqq M_a \leqq \infty, \text{ where } 0 < a < \infty.$$

In fact, it is clear from (2) that it is immaterial whether N_a is referred to $F(t) = F_a(t)$ or to $G(t) = G_a(t)$. On the other hand, a partial integration of (3) shows that $G(t)/t$ is identical with (5) if $\psi(s) = \psi_a(s)$ is defined by (6). This proves the assertion of the last formula line.

Another straightforward consequence (which is related to, but does not involve, the inequality $N \leqq M$) is the complete monotony of $G(t)t = G_a(t)/t$ on $(0,\infty)$ if $0 < a \leqq 1$ (the limiting case $a = 1$ is trivial, since (2) can be evaluated explicitly if $a = 1$). In fact, the substitution $u^{1/a} = v$ shows that the function (6) is non-negative for $0 \leqq s < \infty$ if $0 < a < 1$ (simply because v^{a-1} is positive and non-increasing for $0 < v < \infty$ if $0 < a < 1$). But if $\psi \geqq 0$ throughout, then the function (5) is completely monotone on the half-line $0 < t < \infty$. Since $G(t)/t$ is identical with the case (6) of (5), the assertion follows.

A corollary of this fact is that the function

$$F(1/t)/t^{1+2a}, \text{ where } 0 < t < \infty, \qquad (F = F_a),$$

possesses a Hausdorff-Bernstein representation if $0 < a < 1$ (and, by continuity, also in the limiting case $a = 1$ which, however, is a trivial case). In fact, if $0 < t < \infty$, $0 < a < 1$ and $m = 1, 2, \cdots$, then the m-th derivative of t^a is positive or negative according as m is odd or even. Hence, successive differentiations show that if a function $g(t)$ is completely monotone on $(0,\infty)$, then the same is true of the function $g(t^a)$, where $0 < a < 1$. Hence, the assertion follows by choosing $g(t) = G(t)/t$, since (2) shows that $G(t^a)/t^a$ is identical with the function occurring in the last formula line. (For an application of the substitution $t \to t^a$ to $g(t) = e^{-t}$, instead of the more elaborate function $g(t) = G(t)/t$, cf. (19)-(21) below.)

If $a = 1$, then (1) and (3) show that both $F(t)$ and $G(t)$ reduce to $(1+t^2)^{-1}$ (which argrees with, though it is not implied by, (2) if $a = 1$). Hence the limiting case, $a = 1$, of the preceding result on $G(t)/t$ means that $(t+t^3)^{-1}$ is completely monotone on $(0,\infty)$. The same cannot be true of

$G(t)$ itself, since already the first derivative of $(1+t^2)^{-1}$ fails to be of constant sign on $(0,\infty)$.

5. If t is complex, then the integral (3) is absolutely convergent to the right, and divergent to the left, of the imaginary axis of the t-plane, whenever $0<a<\infty$. But if the point $t=0$ is excluded when $a\geqq 1$, then analytic continuation of $G(t)$ is possible and does not lead to any singularity, and the exclusion of point $t=0$ is unnecessary when $0<a<1$. In fact, it was shown in [13] that if $0<a<1$, then $G(t)$ is a transcendental entire function, having MacLaurin expansion

$$(7)\qquad G(t)=\sum_{n=1}^{\infty} c_{n-1}t^{n-1}/\Gamma(n), \qquad c_{n-1}=(-1)^{n-1}a\Gamma(an)\sin(\tfrac{1}{2}\pi an).$$

The radius of convergence of the power series (7) is ∞, 1 or 0 according as $0<a<1$, $a=1$ or $1<a<\infty$. In the third case, the series (7) is an asymptotic expansion of $G(t)$ for $t\to+0$ which, in view of (2), supplies an asymptotic expansion of $F(t)$ for $t\to+\infty$. In the second case, (7) reduces to $G(t)=(1+t^2)^{-1}$, since c_{n-1} becomes $(-1)^{n-1}\Gamma(n)(-1)^n=-\Gamma(n)$ or 0 according as n is odd or even. In all three cases, the functional equation

$$(8)\qquad \Gamma(1+z)\Gamma(1-z)=\pi z/\sin\pi z$$

leads to some reduction of (7).

As pointed out at the end of [13], the function (3) and its expansion (7) are "formally related to the standard entire functions occurring in the theories of explicit analytic continuations beyond the circle of convergence of a power series," namely, to Mittag-Leffler's transcendents E. But neither (1)-(3) nor (7) is mentioned in the subsequent literature, listed in [1], where [13] is overlooked, as is [14]. Cf. also the footnote at the end of Section 7 below.

6. An approach to (1)-(3) and (7) which is different from that followed in [13] was found in [14], pp. 705-707; it was shown there that the true source of the expansion (7) of (3) is a formula of Mellin [8], p. 12 (which I did not find quoted in the textbooks consulted or in the relevant papers, such as the fundamental paper of Hardy and Littlewood [3], p. 136; cf. p. 120, footnote, which refers to Mellin's work as a whole, but not to his paper [8]; cf. also Lindelöf's general theory in [7]). The formula in

question is an "explicit formula" (in the sense of the term customary in the analytic theory of numbers); it states that

$$(9) \qquad t^{1/a}\sum_{k=1}^{\infty} e^{-tk^a} = \Gamma(1+1/a) + \sum_{n=1}^{\infty} (-1)^n \zeta(-an)\, t^n/\Gamma(n+1),$$

where the numerical value of the leading term on the right is produced, of course, by the trivial identity

$$(9\,\text{bis}) \qquad t^{1/a}\int_0^{\infty} e^{-tx^a}\, dx = \Gamma(1+1/a),$$

and where (in order to assure convergence in the entire t-plane, rather than just the validity of (9) as an asymptotic expansion as $t \to +0$) it is assumed that $0 < a < 1$, rather than only that $0 < a \neq 2, 4, \cdots$. As shown in [14], pp. 705-707, the expansions (7) and (9) are equivalent by virtue of the functional equations (8) and

$$(10) \qquad \eta(z) = \eta(1-z), \text{ where } \eta(z) = \pi^{-\frac{1}{2}z}\Gamma(\tfrac{1}{2}z)\zeta(z).$$

and of a Möbius inversion.

Actually, (9) leads not only to the expansion (7) of $G(t)$ but also to the Laplace analysis of $F(t)$, exhibited by (12) (and (13), (11)) below. In fact, (12) below and (7) are equivalent by virtue of (1)-(3), as seen by a term-by-term integration (the legitimacy of which is trivial from $0 < a < 1$). This is precisely the route along which the expansion (7) was obtained in [13].

7. Let $0 \leqq t < \infty$. Then the Laplace analysis (that is, the determination of the *Unterfunktion*) of $G(t)$ is given by (3), where only $a > 0$ is asumed, and, as mentioned above, (3) follows from the two definitions (1), (2) by nothing deeper than a partial integration. In contrast, the Laplace analysis of $F(t)$ results only after a contour integration, which succeedss directly only if

$$(11) \qquad 0 < a < 1.$$

The result is

$$(12) \qquad F(t) = \int_0^{\infty} e^{-ts} \exp(-\lambda s^a) \sin(\mu s^a)\, ds, \qquad (0 \leqq t < \infty)$$

where $\lambda = \lambda(a)$, $\mu = \mu(a)$ are abbreviations for the constants

$$(13) \qquad \lambda = \cos \tfrac{1}{2}\pi a, \qquad \mu = \sin \tfrac{1}{2}\pi a, \qquad (\lambda^2 + \mu^2 = 1).$$

It is clear from (11) that $t=0$ is the abscissa of convergence, as well as the abscissa of absolute convergence, of the Laplace transform (12). Incidentally, (11) also implies that both constants (13) are positive.

The Laplace analysis (12) of $F(t)$ is known; in fact, it is contained in what was obtained in [12], pp. 86-88, for the m-dimensional generalization (from $m=1$) of the "symmetric stable frequency function" of Cauchy which, if $m=1$, becomes $1/\pi$ times the function (1) itself (it may be mentioned that subsequently Lévy's monograph [6], pp. 221-224, and a dissertation written under his direction (cf. p. 221, footnote) have transferred his fundamental enumeration (1923) of all, not necessarily symmetric, stable frequency functions from $m=1$ to any m; but the Laplace analysis of these functions is not taken up *loc. cit.*, not even in the symmetric case).

The proof of (12) can be sketched as follows (for details, cf. [12], pp. 86-88): In view of (1), where t can be thought of as a fixed positive number, it is sufficient to deal with the real part of the integral

$$\int_0^\infty u^{-1+1/\alpha}\exp(-u+itu^{1/\alpha})\,du, \tag{14}$$

where $u=s^{1/\alpha}\geqq 0$. If $L(\phi)$ denotes, for a fixed ϕ, the half-line $\arg u=\phi$ in the complex u-plane, then $L(0)$ is the path of integration in (14). But no singularity of the integrand is met when $L(\phi)$ sweeps through the wedge $0\leqq\phi\leqq\frac{1}{2}\pi\alpha$ $(<\frac{1}{2}\pi)$, except for the common end-point, $u=0$, which is a harmless singularity (in fact, $-1+1/\alpha>0$). Hence, if $L(0)$ is deformed into $L(\frac{1}{2}\pi\alpha)$, then (12) follows from (11) and (13), simply by taking the real part of the $L(\frac{1}{2}\pi\alpha)$-representation of (14).*

8. From the Laplace analysis, (12), of the Fourier cosine transform, (1), of e^{-t^α}, where $0\leqq t<\infty$, the Stieltjes analysis of e^{-t^α} itself can be

* According to [1], it was conjectured by W. Feller and proved by H. Pollard that $E(-t)$, where $E(z)=E_\alpha(z)$ denotes Mittag-Leffler's entire function $\sum z^n/\Gamma(\alpha n+1)$, is completely monotone on the half-line $0\leqq t<\infty$ if the index α is on the range (11) (this is true, but trivial, in the limiting cases, $\alpha=0$ and $\alpha=1$, of (11), since $E_0(-t)=(1+t)^{-1}$ and $E_1(-t)=e^{-t}$). What is involved here are two papers of Pollard [10] which appeared as late as 1946 and 1948, and which will be referred to as (i) and (ii) respectively. Both (i) and (ii) overlook [12], pp. 86-88, and [13] (cf. also [14], pp. 705-707). But the content of (i) is just an artificial verification of a wrong form of (6) and/or (7) above (in order to see that there is a mistake somewhere, it is sufficient to choose Pollard's λ, which is the α in (11) above, to be $\frac{1}{2}$ in formulae (5) and (4) of (i), which correspond to (7) and (12) above). Correspondingly, the complete monotony of Mittag-Leffler's $E(-t)$ is concluded in (ii) from erroneous formulae. The necessary corrections can be read off from [13] and/or [14].

obtained. In other words, it is possible to obtain the explicit form of the function $h(s)$ for which

$$e^{-t^a} = \int_0^\infty (t^2 + s^2)^{-1} h(s)\, ds \tag{15}$$

holds as an identity for $0 \leqq t < \infty$ (the traditional representation of the Stieltjes transform results from the integral (15) after the substitution $(t, s) \to (t^{\frac{1}{2}}, s^{\frac{1}{2}})$ and a renaming of f).

First, Fourier inversion of (1) gives

$$\tfrac{1}{2}\pi e^{-t^a} = \int_0^\infty F(x) \cos tx\, dx \qquad (0 \leqq t < \infty). \tag{16}$$

Next, if (12) is inserted in (16), the order of the integrations can be interchanged (by uniform convergence, if $t > 0$). Since

$$\int_0^\infty e^{-tx} \cos sx\, dx = t/(t^2 + s^2) \tag{17}$$

(if $t > 0$), it follows that (15) will be satisfied if (and, in view of the uniqueness theorem of Stieltjes' transform, only if) the function $h(s)$ is chosen as follows:

$$\tfrac{1}{2}\pi h(s) = s \sin(\mu s^a) \exp(-\lambda s^a). \tag{18}$$

From the *Unterfunktion*, (18), of the Stieltjes analysis, (15), of the function e^{-t^a} the *Unterfunktion*, say $f(s)$ of its Laplace analysis

$$e^{-t^a} = \int_0^\infty e^{-ts} f(s)\, ds, \tag{19}$$

where $0 \leqq t < \infty$, can be obtained, with the following result: If the constants λ, μ are defined by (13), and if (11) is assumed, then the f for which (19) holds for $0 \leqq t < \infty$ as an identity is given by

$$\tfrac{1}{2}\pi f(s) = \int_0^\infty e^{-\lambda x^a} \sin \mu x^a \sin sx\, dx, \tag{20}$$

It may be mentioned that, while (11), (13) and (20) hardly indicate that

$$f(s) \geqq 0 \text{ for } 0 \leqq s < \infty, \tag{21}$$

(21) happens to be true. In fact, successive differentiations of the function e^{-t^α} readily shows that, if $0 < t < \infty$ and $0 < \alpha < 1$, its n-th derivative is positive or negative according as n is even or odd. Hence the Hausdorff-Bernstein theorem on completely monotone functions, when combined with the uniqueness theorems of Laplace's transform, implies that (21) is a consequence of (19).

9. The deductions of (20) from (18) has hardly anything to do with the explicit form of the function e^{-t^α}; it depends merely on the following formal rule: If

$$\int_0^\infty (t^2 + s^2)^{-1} h(s)\, ds = \int_0^\infty e^{-ts} f(s)\, ds \tag{22}$$

is an identity for $0 \leqq t < \infty$, then (under appropriate conditions, conditions which are amply satisfied in the case of e^{-t^α}) the connection between f and h is given by Dirichlet's transform,

$$f(s) = \int_0^\infty h(x) \sin sx / x\, dx. \qquad (0 < s < \infty). \tag{23}$$

Hence (20) follows from (18). (Incidentally, if f, rather than h, is known in (22), then, subject to the legitimacy of the Fourier inversion which is involved,

$$\tfrac{1}{2}\pi h(s)/s = \int_0^\infty f(x) \sin sx\, dx, \qquad (0 < s < \infty); \tag{24}$$

in fact, (23) is the Fourier sine transform of $h(s)/s$, and so (24) follows by Fourier reciprocity).

I cannot decide whether the solution (23) of (22) is generally known; in view of [2], something like (22)-(23) must have been familiar to Ramanujan. In any case, it would be worthwhile to exploit this rule systematically, *by completing the tables of known Laplace transforms (for a given* Unterfunktion f*) by those items for which the* Unterfunktion h *of the Stieltjes transform is already recorded in the tables of the latter transform.* The actual validity of the Dirichlet rule (23) in the appropriate Hilbert space (that is, the specification of such function spaces) can be justified along the lines of the Plancherel technique of Paley and Wiener [9] (*passim*; e.g., pp. 42-43). In such cases as e^{-t^α}, neither locally nor in Hilbert's space is there an issue, and (20) could be deduced from (15) by an appeal to

Fubini's theorem. But it is somewhat more instructive to verify (22) under the drastic hypothesis that it is legitimate "to compare like powers."

To this end, note that, if $(t^2+s^2)^{-1}$ is expanded according to powers of $1/t$, then, after a term-by-term integration (if it is allowed), the integral on the left of (22) appears in the form

$$\sum_{n=0}^{\infty}(-1)^n\mu_{2n}(h)/t^{2(n+1)}, \text{ where } \mu_n(h)=\int_0^{\infty}s^nh(s)\,ds.$$

On the other hand, successive partial integrations lead to the following expansions of the integral on the right of (22):

$$\sum_{n=0}^{\infty}f^{(n)}(0)/t^{n+1}, \text{ where } f^{(n)}=d^nf/ds^n$$

(if $f(s)$ is of class C^{∞}). Hence, comparison of the respective powers of $1/t$ (if it is allowed) shows that

$$f^{(2n)}(0)=0 \text{ and } f^{(2n+1)}(0)=(-1)^n\int_0^{\infty}s^{2n}h(s)\,ds,$$

where $n=0,1,\cdots$. Thus, if $f(s)=\sum_{n=0}^{\infty}f^{(n)}(0)s^n/n!$, it follows that

$$f(s)=\sum_{n=0}^{\infty}(-1)^ns^{2n+1}/(2n+1)!\int_0^{\infty}x^{2n}h(x)\,dx,$$

which is (23) if $\Sigma\int=\int\Sigma$.

10. The suggestion italicized above, concerning the explicit formulation (23) (or, in the reverse direction, (24)) of the hypothesis (22), is illustrated by the transition from (15) and (18) to (19) and (20). In view of (17), this can be re-stated so as to replace the Stieltjes transform by a Fourier *cosine* transform (whereas (23) or (24) is a Fourier *sine* transform); in fact, the e^{-x^a} in (1) and (16) can be replaced by any function which is "small and smooth." Actually, all of this can be formulated as a general principle, concerning the translation of one transform table into another transform table, as follows:

For $M=K,M,N,\cdots$, let $M=M(t,s)$ be the kernel of an integral transformation

$$\int_0^{\infty}M(t,s)g(s)\,ds \tag{25}$$

which sends "arbitrary" functions g on $0 < s < \infty$ into "arbitrary" functions $g^* = Mg$ on $0 < t < \infty$ and which has the property that (in terms of an appropriate definition of identity) the *M-Unterfunktion*, $M^{-1}g^*$, of g is unique. With reference to two such transformations, M and N, let there be known a $K = K_{MN}$ for which $M^{-1} = KN^{-1}$ holds (in view of the formulation (23) of (22), this is the case if $M = L$, $N = S$ and $K = D$, where L, S and D denote the transforms of Laplace, Stieltjes and Dirichlet respectively, with (t, s) replaced by (t^2, s^2) in the traditional representation of S). Then, if the *N-Unterfunktion* of a *given* function of t happens to be known explicitly, the *N-Unterfunktion* of the *same* function of t follows by an integration, that assigned by K.

Instances and general methods are contained in [2] and [9].*

Let the function (25) of t on $(0,\infty)$ be denoted by $S_t f$, $L_t f$, $C_t f$ in the respective cases $M = S, L, C$, where

$$(26_S) \quad S = (t^2 + s^2)^{-1}; \qquad (26_L) \quad L = e^{-ts}; \qquad (26_C) \quad C = \cos ts.$$

Note that D is the integral (vanishing at the origin) of C; cf. (23).

By Fourier's inversion, $\frac{1}{2}\pi C^{-1} = C$ (formally; so that, in particular, (17) is equivalent to

$$(17 \text{ bis}) \qquad \int_0^\infty (t^2 + x^2)^{-1} \cos sx \, dx = \tfrac{1}{2}\pi e^{-ts}/t,$$

where $t \neq 0$.) It follows that if

$$(27) \qquad \int_0^\infty |f(s)| \, ds < \infty,$$

then, for $0 < t < \infty$,

$$(28) \qquad (LC)_t f = S_t f_1, \text{ where } f_1(s) = f(s)s,$$

and that (28) has the dual

$$(28 \text{ bis}) \qquad (SC)_t f = \tfrac{1}{2}\pi L_t f_{-1}, \text{ where } f_{-1}(s) = f(s)/s,$$

* For specific pairs M, N of certain type, conditions under which this obvious remark is legitimate (for a given f) can be read off, for instance, from such considerations as were collected by Hirschman and Widder in their book [5] (in which, incidentally, it is again overlooked, as in the other publications of these authors and in those of I. J. Schoenberg which they quote, that the Fourier representation of the reciprocal Weierstrass products was studied already in [12], pp. 51-54 and 61-64, where, in the even case, these functions are introduced precisely in the same way as then followed by I. J. Schoenberg, cf. [5]).

if, in addition to (27),

$$\int_{+0} |f(s)/s|\,ds < \infty. \tag{27 bis}$$

In fact, the content of (28) has already been used. But in view of (17) and its Fourier inversion (17 bis), both (28) and (28 bis) follow from Fubini's theorem. As mentioned above, there is a corresponding deduction of (23) from (22). It is also clear that, under the respective Fubini assumptions, (27) or (27) *and* (27 bis), of (28) and (28 bis),

$$LC = CL \text{ in (28), and } SC = CS \text{ in (28 bis).} \tag{29}$$

The C-duality, (28)-(28 bis) and (29), of L and S has an analogue, but becomes involutory, if (26_C) is retained but (26_L) is replaced by the "normal" (Gaussian) kernel

$$N = e^{-t^2 s}. \tag{26_N}$$

What then corresponds to (26_S) is, in the main, (26_N) itself, since what corresponds to (17) *and* to (17 bis) is contained in the *single* relation

$$\int_0^\infty e^{-x^2} \cos yx\,dx = \tfrac{1}{2}\pi^{\frac{1}{2}} e^{-z^2}, \text{ where } z = \tfrac{1}{2}y. \tag{30}$$

While this parallelism is trivial, it turns out that, owing to the (N,N)-analogue of the (L,S)-rules (28)-(29), the explicit results (16)-(20), where (11) and (13) are assumed, settle a desideratum I formulated a long time ago, concerning the actual determination of the weight functions which, *for all values of* β *on the range* $0 < \beta < 2$, are the factors of normal stratification ("Gaussian analysis") of Cauchy's (symmetric) *stable distribution functions of index* β; cf. [4], Section 4, where further references are given (p. 769, footnote [20]).

11. In order to avoid a confusion of the index α, occurring in (11) and (13), with Cauchy's index β, let the functions F, G, h, f, occurring in (1)-(3) and (18)-(20), be now denoted by F_α, G_α, h_α, f_α, and let $\beta = \beta(\alpha)$ be an abbreviation for

$$\beta = 2\alpha; \text{ so that } 0 < \beta < 2, \tag{31}$$

by (11). Thus F_γ, G_γ are defined, and (1)-(3) and (16) remain valid, if $\gamma = \alpha$ is replaced by $\gamma = \beta$, but (7), (9), (12), (15) and (19)-(20) then

become divergent (and, incidentally, the analogue of (21) becomes wrong). The notation $\lambda = \lambda(\gamma)$, $\mu = \mu(\gamma)$ will always be used only with reference to $\gamma = \alpha$ (so that the numbers (11) will remain positive).

Except for a trivial factor $(= 1/\pi)$, Cauchy's (symmetric) "stable frequency functions" of index β is $F_\beta(t)$, which means, by (1), that

$$F_\beta(t) = \int_0^\infty e^{-s^\beta} \cos ts \, ds. \tag{32}$$

On the other hand, if the t in (19), where $0 \leqq t < \infty$ and $0 < \alpha < 1$, is replaced by t^2, it follows from (31) that

$$e^{-t^\beta} = \int_0^\infty e^{-t^2 s} f_\alpha(s) \, ds. \tag{33}$$

It follows therefore from (32) (and from Fubini's theorem) that

$$F_\beta(t) = \int_0^\infty f_\alpha(s) \{ \int_0^\infty e^{-x^2 s} \cos tx \, dx \} ds.$$

But here the $\{\ \}$ is $\frac{1}{2}\pi^{\frac{1}{2}} \exp(-t^2/4s^2)$, as seen by changing the integration variable in (30). Accordingly

$$F_\beta(t) = \tfrac{1}{2}\pi^{\frac{1}{2}} \int_0^\infty \exp(-t^2/4s^2) f_\alpha(s) \, ds. \tag{34}$$

The relation (34) is the explicit form of the stratification, announced above. In fact, the weight function occurring in (34) is

$$f_\alpha(s) = 2\pi^{-1} \int_0^\infty \sin sx \, e^{-\lambda x^\alpha} \sin \mu x^\alpha \, dx \geqq 0, \quad \text{cf. (11)}, \tag{35}$$

by (20) and (21). Incidentally, the inequality in (35) makes it clear that the function (34) of t^2 is not only positive but decreasing as well. This property of Cauchy's stable densities $F_\beta(t)$, where $0 < \beta < 2$, is significant from the point of view of the theory of probability (cf. [12], pp. 70-71); it was proved in [12], pp. 83-86, by arguments having the nature of an existence proof, instead of being explicit, as in (34)-(35).

Another "stratified" representation of $F_\beta(t)$ (which, however, fails to exhibit the essential information $f_\alpha \geqq 0$ in (34)) is the following:

$$F_\beta(t) = \pi^{-\frac{1}{2}} \int_0^\infty E(\tfrac{1}{2}t, x) \operatorname{Im} \exp(-\lambda + i\mu) x^\alpha \, dx, \tag{36}$$

where $\lambda = \lambda_\beta > 0$, $\mu = \mu_\beta > 0$ and $a = a_\beta < 1$ are defined by (11) and (31), and $E(t, s)$ is the function

$$E(t, x) = \operatorname{Im} \int_0^\infty \exp(ixs - t^2/s^2)\, ds, \tag{37}$$

which is independent of β (< 2). In fact, if (35) is inserted in (34) and the order of integrations is interchanged (which can readily be justified), then (36) follows from (37).

12. If $F(t)$ is any *symmetric frequency curve* $(= F(-t) \geqq 0$, where $-\infty < t < \infty)$, it is called *unimodal* if $F(t)$ is a non-increasing function of $t > 0$, and it is called *bell-shaped* if it is such that there exists a (not necessarily unique) positive t_0 having the following property: The curve $F = F(t)$ is concave or convex (toward the t-axis) according as $0 \leqq t \leqq t_0$ or $t_0 \leqq t < \infty$. Both of these classes of frequency curves F, the second of which is a subclass of the first, can be defined without the assumption $F(t) = F(-t)$ also. The unimodal character of (34), just concluded from the inequality in (35), can of course be concluded also if (34) is generalized to

$$F(t) = \int_0^\infty \exp(-t^2 x)\, d\phi(x), \tag{38}$$

$(x = 1/4s^2)$, where $\phi(x)$ is any function satisfying

$$\int_0^\infty d\phi(x) = \int_0^\infty |\, d\phi(x)\,| < \infty, \text{ i.e., } d\phi(x) \geqq 0,\ \phi(\infty) - \phi(0) < \infty. \tag{39}$$

The Gaussian limiting case, $\beta = 2$, of (34) results from (36) if $\phi(x)$ is chosen to be the step-function $\phi_{pa}(x) = p \operatorname{sgn}(x - a)$, where p and a are positive constants. But whereas (38) is bell-shaped if $\phi = \phi_{pa}$, it is possible to choose four positive constants p, a; q, b in such a way that the case $\phi = \phi_{pa} + \phi_{qb}$ of (38), the superposition of two symmetric normal frequency curves, fails to become bell-shaped; cf. [11]. Thus, whereas one might hope that (34) is bell-shaped not only in the limiting case $\beta = 2$ but for every positive $\beta < 2$, this, if true, cannot be concluded from nothing more than the inequality in (35).

If $\beta = 1$, then Cauchy's frequency function (34) is known to reduce to the elementary function $(1 + t^2)^{-1}$ (except for constant factors) and is therefore bell-shaped. Since this means that $d^2F_\beta(t)/dt^2$ has a (unique)

positive zero $t_0 = t_0(\beta)$ if $\beta = 1$, it can be concluded from (34) and from the explicit form (35) of $d\phi(x)/dx$ that such a $t_0(\beta) > 0$ exists not only for $\beta = 1$ but for every β contained in the range $1 - \epsilon < \beta < 1 + \epsilon$, where $\epsilon > 0$ is a sufficiently small explicit constant. But even this continuity argument can be justified only by a certain amount of explicit labor, and the desideratum (namely, that the entire range, $0 < \beta < 2$, of Paul Lévy can be included, so that ϵ can be chosen to be 1) cannot apparently be obtained along these lines.*

13. Let a in (1) now be any positive index (not restricted by $0 < a < 1$ or $0 < a < 2$). Then (4) is still valid, as is (3) with (2). If $a > 1$, then the series (7) is divergent at every $t > 0$. But it can be seen from the proof of (7) that (7) holds, as $t \to +0$, as an asymptotic development,

$$G(t) \sim \sum_{k=0}^{\infty} c_{k-1} t^{k-1}/\Gamma(k), \qquad c_k = (-1)^{k-1} a\Gamma(ak) \sin(\tfrac{1}{2}\pi ak). \tag{41}$$

In view of (2), this is an elaboration of (4), where $t \to \infty$. If a is an even integer, $a = 2n$, then all coefficients of (41) become 0, hence (41) means that, for every fixed $\epsilon > 0$,

$$G(t) = O(t^{1/\epsilon}) \text{ as } t \to 0 \qquad (a = 2n) \tag{42}$$

and so, in view of (2),

$$F(t) = O(t^{-1/\epsilon}) \text{ as } t \to \infty \qquad (a = 2n) \tag{43}$$

(actually, the by-product (42) of (41) can be obtained directly; in fact, it is readily seen from (3) that all derivatives of the function $G(t)$, which clearly is analytic for $0 < t < \infty$, tend to 0 as $t \to 0$, if $t > 0$).

If $2n = 2$, then (43) is obvious, since (1) reduces to

$$F(t) = \tfrac{1}{2}\pi^{\frac{1}{2}} \exp(-t^2/4) \text{ if } a = 2.$$

---

* It is known that if $\Phi(x) = \Phi(-x)$, where $-\infty < x < \infty$, denotes the Fourier transform of Riemann's $\Xi(t) = \xi(s) = \Xi(-t)$, where $s = \frac{1}{2} + it$, then not only $\Phi(x) > 0$ holds (Jensen, Hurwitz) but also $d\Phi(x)/dx < 0$, where $0 < x < \infty$ (see my note in the Journ. London Math. Soc., vol. 10 (1935), pp. 82-83). Is it also true that $\Phi(x)$ is bell-shaped?

Incidentally, it is clear from the two known properties of $\Phi(x)$ that the first of the functions

$$\int_0^{\infty} \Phi(x) \sin tx\, dx, \qquad \int_0^{\infty} \Phi(x) \cos tx\, dx \tag{40}$$

is positive for $0 < t < \infty$, whereas the second changes sign an infinity of times, since $\Xi(t)$ does.

One will therefore expect that, if

$$\alpha = 2n, \text{ where } 2n = 4, 6, 8, \cdots, \tag{44}$$

then $F(t)$ will be much smaller than the order supplied by (43). But after the exclusion of the trivial case $2n = 2$, the function $F(t) = F_{2n}(t)$ changes sign an infinity of times as $t \to \infty$ (in this regard, cf. the known results of Pólya, referred to at the end of Section 2), and what one will conjecture for (44) is the existence of an explicit asymptotic formula

$$F(t) \sim f(t) \cos g(t) \text{ as } t \to \infty, \tag{45}$$

where $f(t)$ and $g(t)$ are elementary functions which satisfy $-\log\log f(t) \to \text{Const.} > 0$ and $\log g(t) \to \text{const.} > 0$, where $t \to \infty$ (and where Const. and const. depend on the index (44)). In a letter to Professor Pólya, I raised this question a long time ago, mentioning the analogy between the conjectural formula (45) and the known formulae for the "(real) definite integral" solutions of the confluent forms of hypergeometric equation (Bessel-Poisson; Laguerre-Perron); the case $\alpha = 2n$ of being a "Laplace" solution of a simple differential equation, of order $n-1$ which is homogeneous and linear:

$$d^{n-1}F(t)/dt^{n-1} + atF(t) = 0, \text{ where } a = a_n = \text{const.} = (-1)^{n-1} \mid a_n \mid^{n}. \tag{46}$$

In another context (not that of Cauchy's "stable densities"), the identity (46) for the case $\alpha = 2n$ of (1), readily verified by successive partial integrations, is known since Jacobi's time (at least); cf., e.g., p. 152 of E. G. C. Poole's *Linear Differential Equations* (Oxford, 1936).

Professor Pólya was good enough to determine the explicit form of (45) and to communicate it to me in a letter dated November 18, 1944, the content of which is only now published on his request. The formula is of an impressive length; I shall write it in terms of notations corresponding to those used in (12) (where (11), rather (44), is the assumption).

In terms of a fixed n (> 1), put $\nu = (n-1)/(2n-1)$, $\kappa = 2n/(2n-1)$ (hence $\nu > 0$) and

$$\lambda = -\rho \cos(\tfrac{1}{2}\pi\kappa), \quad \mu = -\rho \sin(\tfrac{1}{2}\pi\kappa), \text{ where } \rho = (2n-1)/(2n)$$

(so that $\lambda > 0$, since, in view of (44), the angle $\frac{1}{2}\pi\kappa = \pi n/(2n-1)$ is between $\frac{1}{2}\pi$ and π). Then, if $\alpha = 2n$ and $t \to \infty$ in (1),

$$F(t) \sim Ct^{-\nu} \exp(-\lambda t^{\kappa}) \cos(\mu t^{\kappa} + \tfrac{1}{2}\pi\nu),$$

where $C = (8\pi/c)^{\frac{1}{2}}$, $c = m(1+m)^{1/m}$, $m = 2n-1$.

Pólya added in his letter that he could foresee this structure of $f(t)$ and $g(t)$ in (45) from the order of magnitude of purely imaginary values, which can be ascertained from the Maclaurin series of (1) and from (46); and that he obtained the final result by writing the case (44) of (1) in the form

$$\text{(14 bis)} \qquad \int_{-\infty}^{\infty} \exp\{itx - x^{2n}\}dx \equiv t^{1/m} \int_{-\infty}^{\infty} \exp\{(ix - x^{2n})t^{2n/m}\}dx,$$

where $m = 2n-1$ and $0 < t < \infty$, and applying the method of steepest descent to the last integral.

APPENDIX I.*

Reduction mod 1.

Let $0 < a < \infty$, $0 < q < 1$, $-\infty < x < \infty$, and put

$$\text{(1)} \qquad \theta_a(x;q) = 1 + 2\sum_{m=1}^{\infty} q^{m^a} \cos 2\pi mx;$$

so that $0 \leqq x < 1$ without loss of generality. This is Poisson's kernel if $a = 1$, and an elliptic theta function if $a = 2$.

If $a = 2$, then the function

$$\text{(2)} \qquad F_a(t) = \int_0^{\infty} \exp(-s^a) \cos ts\, ds = F_a(-t) \qquad (0 \leqq t < \infty)$$

differs from its Fourier reciprocal (given by

$$\text{(3)} \qquad \pi \exp(-|t|^a) = \int_0^{\infty} F_a(s) \cos ts\, ds$$

for any $a > 0$) only in two constant factors. Hence, when the functional equation (the "linear transformation") of the case $a = 2$ of (1) is proved by means of Poisson's summation formula

$$\text{(4)} \qquad \sum_{k=-\infty}^{\infty} \phi(x+k) = \sum_{m=-\infty}^{\infty} e^{2\pi imx} \int_{-\infty}^{\infty} \phi(y) e^{-2\pi imy}\, dy,$$

it is immaterial whether $\phi(rt)$, where r is any positive constant, is chosen

* Added July 21, 1956.

to be the function (2) or the function (3). On the other hand, the two choices lead to two different identities in every non-involutory case ($a \neq 2$).

The first choice of ϕ leads to the following representation of (1) (for any $a > 0$):

$$2\pi\theta_a(x;q) = p \sum_{k=-\infty}^{\infty} F_a(px + pk), \tag{5}$$

where $p = p(q) = p_a(q)$ is defined by

$$(-\log q)^{1/a} = 2\pi/p \tag{6}$$

(so that the range $0 < q < 1$ is mapped on the half-line $0 < p < \infty$). In fact, if $\phi(x) = F_a(px)$, where $-\infty < x < \infty$ and $p = \text{const.} > 0$, then, since (2) is equivalent to

$$2\pi p^{-1} \exp(-|x|^a/p^a) = \int_{-\infty}^{\infty} F_a(py) e^{-ixy}\, dy,$$

the series on the right of (4) becomes $2\pi p^{-1} \sum e^{2\pi i m x} \exp(-|2\pi m|^a/p^a)$, which is $2\pi p^{-1}\theta_a(x;q)$, by (1) and (6). Hence (5) follows by inserting $\phi(x) = F_a(px)$ on the left of (4) also.

If a is of the form $2n$, then (2) is majorized by $\exp(-a|t|^b)$, where $a = a(a) > 0$ and $b = b(a) > 0$ (cf. Section 13), and so it is clear that

$$\sum_{k=-\infty}^{\infty} F_a(px + pk) \sim F_a(px) \text{ as } p \to \infty \tag{7}$$

holds at every fixed $x > 0$, and uniformly for $\epsilon < x < 1/\epsilon$ if $\epsilon > 0$ is fixed. It follows therefore from (5) and (6) that, for $0 < x < 1$, and uniformly for $\epsilon < x < 1 - \epsilon\ (< 1)$,

$$\theta_a(x;q) \sim |\log q|^{-1/a} F_a(2\pi |\log q|^{-1/a} x) \text{ as } q \to 1. \tag{8}$$

In the classical case, where $a = 2$, the asymptotic formula (8) becomes dull, since $F_2(t) = \pi^{-\frac{1}{2}} \exp(-t^2/4)$ is free of fluctuations. But if $a = 2n > 2$, then, by Pólya's formula,

$$F_a(t) \sim Ct^{-\nu} \exp(-\lambda t^\kappa) \cos(\mu t^\kappa + \tfrac{1}{2}\pi\nu) \text{ as } t \to \infty \tag{9}$$

holds for the (even) function (2), and C and the four Greek indices occurring on the right of (9) are certain (non-vanishing) functions of n (> 1) alone (cf. Section 13), and (8) and (9) (where $\lambda > 0$ and $\nu > 0$) show that, as $q \to 1$, the function (1) of q is wobbly at every x distinct from 0 (mod 1), and uniformly for $\epsilon < x < 1 - \epsilon\ (< 1)$.

If a is not an even integer, then the asymptotic behavior ($q \to 1$) of

(1) is as dull as it is in the case $\alpha = 2$. But (8) is then wrong, since the terms of the series (5) are damped (for increasing $|k|$) with sufficient rapidity only if $\alpha = 2n$ (where $\alpha = 1$ is allowed). In fact, if $0 < \alpha < \infty$ but $\alpha \neq 2n$, then $F_\alpha(t) = F_\alpha(-t)$ is asymptotic to $A/|t|^{1+\alpha}$ as $t \to \infty$, where $A = A(\alpha)$ is a non-vanishing constant (cf. formula (4) of Section 2). Hence, if $p \to \infty$, then, for fixed $x > 0$ (and uniformly for $\epsilon < x < 1/\epsilon$), the function on the left of (7) is asymptotic to $A\psi_\alpha(x)/p^{1+\alpha}$, where

$$(10^*) \qquad \psi_\alpha(x) = \sum_{k=-\infty}^{\infty} |x+k|^{-1-\alpha} = \psi_\alpha(x+1) \qquad (\alpha > 0).$$

It follows therefore from (5) and (6) that, uniformly for $\epsilon < x < 1-\epsilon (< 1)$,

$$(10) \qquad \theta_\alpha(x;q) \sim B\psi_\alpha(x) \log q \text{ as } q \to 1,$$

where $B = -A/(2\pi)^{1+\alpha}$, $A = \Gamma(1+\alpha)\sin(\frac{1}{2}\pi\alpha)$.

If $x = 0$, then, although (5) is valid, (8) is false, and (10) is meaningless (because (10*) includes $k = 0$). But if $x = 0$ and $q \to 1$, then there is no issue, since (1) shows that, if $r = -\log q$ (hence $0 < r \to 0$), then $\frac{1}{2}\theta_\alpha(0;q) - \frac{1}{2}$ is

$$\sum_{m=1}^{\infty} \exp(-rm^\alpha) \sim I_\alpha(r), \text{ where } I_\alpha(r) = \int_0^\infty \exp(-ru^\alpha)\,du,$$

and since $I_\alpha(r) = I_\alpha(1)/r^{1/\alpha}$, where $I_\alpha(1) = \Gamma(1+1/\alpha)$.

Consider finally (1) as a function of x at a fixed q ($\neq 0$; so that the case $\theta_\alpha(q;x) = \text{const.}$ will now be excluded). Clearly, $\theta_\alpha(q;x)$ is of class C^∞ on $0 \leqq x < 1 \pmod 1$ whenever $\alpha > 0$, is analytic at every real x if $\alpha = 1$, and is a transcendental entire function of the complex variable x if $\alpha > 1$. If $\alpha < 1$, then (1) cannot be analytic for all real x, since the coefficient of the m-th cosine fails to be majorized by the m-th term of a convergent geometrical progression. But (1) is analytic on the interval $0 < x < 1$ (with a singularity of the C^∞-type, like $\exp(-x^{-2})$, at $x = 0$) if $\alpha < 1$. This can be seen as follows:

If the $1 + 2\sum$ of (1) is simplified to the sum $\sum$ itself, then what results is the real part of $P(z) = \sum q^{m^\alpha} z^m$ on $|z| = 1$ ($\arg z = 2\pi x$), where $m = 1, 2, \cdots$. Hence the assertion is that this power series $P(z)$ (which, since $\alpha < 1$, diverges if $|z| > 1$) converges for $|z| < 1$ to a function which remains regular at every point $z \neq 1$ of $|z| = 1$. Actually, the analytic continuation of $P(z)$ exists (as a single-valued regular function) on the domain which results if the half-line $1 \leqq z < \infty$ is removed from the z-plane. In fact, the coefficient of z^m in $P(z)$ is of the form $f(m)$, where $f(w)$ is an

entire function which, since $|q|<1$ and $0<a<1$, is $O(\exp|w|^{\epsilon})$ as $|w|\to\infty$, if $\epsilon>0$ is fixed. Hence the assertion follows from a known general theorem (cf. [7], p. 109).

In order to exhibit the precise nature of the singularity of (1) at $x=0$ (if $0<a<1$ and $0<q<1$), let the expansion (7) of Section 5, an expansion valid for all t, be inserted in formula (2) of Section 2. This shows that

$$F_a(t)=\sum_{j=1}^{\infty} b_j/t^{1+aj} \text{ for } 0<t<\infty, \tag{11}$$

where the coefficients $b_j=b_j(a)$ are given by

$$b_j=c_{j-1}/\Gamma(j), \qquad c_{j-1}=(-1)^{j-1}a\Gamma(aj)\sin(\tfrac{1}{2}\pi a). \tag{11 bis}$$

On the other hand, since (6) is positive and (2) is even, (5) can be written in the form

$$2\pi\theta_a(x;q)=p\sum_{k=0}^{\infty}F_a(pk+px)+p\sum_{k=1}^{\infty}F_a(pk-px),$$

if $x\geqq 0$. Hence, if the F_a of the latter two series are substituted from (11) and the summation order is interchanged (which is readily justified, since the coefficients (11 bis) decrease rapidly enough, and since $F_a(t)=O(t^{-1-a})$ as $t\to\infty$), it is seen from (6) that

$$2\pi\theta_a(x;q)=\sum_{j=1}^{\infty} b_j\chi_a{}^j(x)/(-2\pi)^{aj} \tag{12}$$

holds for $0<x<1$ and for every $a<1$, where the functions $\chi_a{}^1(x), \chi_a{}^2(x), \cdots$ denote the following counterparts of the functions (10*):

$$\chi_a{}^j(x)=\sum_{k=0}^{\infty}(k+x)^{-1-aj}+\sum_{k=1}^{\infty}(k-x)^{-1-aj}. \tag{12*}$$

The representation (12) of (1), a representation which in view of (12*) is (in an appropriate sense) of the type of an Eisenstein series (for a modular function), puts into evidence the nature of the singularity acquired by (1) when x tends to 0 (mod 1) if $a<1$. This "decomposition into singularities" was obtained in [19] by the above method but with algebraic errors, corrected in (12) and (12*) above (the errors were introduced in [19] by an erroneous copying (from [13]) of the exponents in (11) above, that is, in the [correct] result of [13]).

Another application of (5) results if use is made of formulae (31)-(35) of Section 11. According to those formulae, which are valid for $F_a(t)$ if $a<2$ (rather than only if $a<1$),

$$\pi^{\frac{1}{2}}F_a(t) = \int_0^\infty \exp(-t^2/4s^2)\phi_a(s)\,ds, \tag{13}$$

if

$$\phi_a(s) = \int_0^\infty \sin(sy)\exp(-\lambda y^{\frac{1}{2}a})\sin(\mu y^{\frac{1}{2}a})\,dy, \tag{14}$$

where, corresponding to the definitions (11) in Section 7, $\lambda = \cos(a\pi/4)$, $\mu = \sin(a\pi/4)$. It is clear from (5) and (13) that

$$2\pi^{\frac{1}{2}}\theta_a(x;q) = \int_0^\infty \phi_a(s)\Theta(x;s)\,ds, \qquad (0 < a < 2), \tag{15}$$

where, if $p = p_a(q)$ is defined by (6),

$$\Theta(x;s) = p\sum_{k=-\infty}^{\infty} \exp\{-(px+pk)^2/4s^2\}. \tag{15*}$$

But the case $a = 2$ of (5) shows that (15*) is of the form $\theta_2(x;r)$, where θ_2 is an elliptic theta and $r = r(s,p) = r_a(s;q)$. Since the function (14) is positive for $0 < s < \infty$ (cf. (20)-(21), Section 8), it follows that (15) is the analogue for "angular distributions" of that "Gaussian stratification" the explicit form of which was obtained in Section 11.

It was Zernike's discussion of angular statistics ([20], pp. 477-478) which led me in [16] to the interpretation of the function (1) as the "densities of stable, symmetric, angular distributions." Subsequently, these considerations were rediscovered, and further developed (also in the unsymmetric case) by Lévy [15]. The proof of a certain property ("symmetrical convexity" mod 1) of the case $a = 2$ of (1) is incomplete in [15], p. 36; a simple proof of that property was given in [17], and a refinement ("bell-shaped graph" mod 1; cf. Section 12 above) of that property in [18]. The existence of (15), without the explicit form, (14), of the weight function of (15), was shown in [17], where the weaker one of the two properties, just mentioned, was extended from $a = 2$ to $0 < a < 2$ (that extension is contained in the fact that the weight function (15) is positive for $0 < s < \infty$).

It should finally be mentioned that there is a fundamental difference between the "linear" and the "angular" case. If the unit of length is suitably chosen, then the stable distributions having an even density are given by (2), where, however, the index $a > 0$ cannot be chosen arbitrarily (as proposed by Cauchy), since the density does not become negative if and only if $a \leqq 2$ (Lévy). In contrast, it is clear that there belongs to *every* a,

where $0 < a < \infty$, a sufficiently small $q^* = q^*_a > 0$ in such a way that the function (1) (in which the value of the parameter q has the role of a standard deviation) is positive for all x whenever $0 \leqq q < q^*$.

APPENDIX II.

On certain unimodal distributions.

In the nomenclature of [12], the result of [12], p. 78, refined to an explicit representation by Section 11 above, was as follows:

(†) *Every symmetric stable distribution* (*Cauchy-Lévy*) *is a convex distribution.*

The stable distributions are the simplest instances of the so-called L-distributions (cf. below). Hence (†) is contained in the following theorem (*), the proof of which is the first purpose of this appendix:

(*) *Every symetric L-distribution* (*Khintchine-Lévy*) *is a convex distribution.*

In their monograph [22], Kolmogoroff and Gnedenko formulate, and attempt to prove, a theoerm which would contain (*). But as observed by K. L. Chung, the proof is based on an erroneous lemma. Cf. the footnote at the end of Appendix III below. Correspondingly, the following proof of (*) will have to be based on a more involved adaptation of the proof of (†) in [12].

Let $F_\phi(t)$, where $-\infty < t < \infty$, denote the Fourier-Stieltjes transform,

$$F_\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, d\phi(x), \tag{1}$$

of a distribution $\phi = \phi(x)$, that is, of a function defined for $-\infty < x < \infty$ in such a way that

$$\phi(-\infty) = 0, \quad \phi(\infty) = 1 \text{ and } d\phi(x) \geqq 0. \tag{2}$$

If $\phi(x)$, when normalized by $2\phi(x) = \phi(x+0) + \phi(x-0)$, is identical with the distribution $1 - \phi(-x)$, then $\phi(x)$ is called symmetric. By a convex distribution is meant a symmetric unimodal distribution, that is, a distribution which for $0 < x < \infty$ and $-\infty < x < 0$ (but necessarily at $x = 0$) has a (not necesarily continuous) density which is a monotone function of $|x|$ $(\neq 0)$.

The proof of (*) will be reduced to the following lemma: If $S(t)$ denotes the (even, entire) function

$$S(t) = \int_0^t (\sin u)^2/u \, du, \tag{3}$$

and if q is any positive constant, then the Fourier inversion of the case

$$F_\phi(t) = e^{-qS(t)} \qquad (-\infty < t < \infty) \tag{4}$$

of (1) defines a unimodal distribution $\phi = \phi_q = \phi_q(x)$. Since (4) is even, hence $\phi(x) = 1 - \phi(-x)$, this means that every $\phi = \phi_q$ is a convex distribution.

It is clear that if $\phi(x)$ is any unimodal distribution, and if p is any positive constant, then $\psi(x) = \phi(px)$ is a unimodal distribution. It is also clear that if $\phi = \phi(x)$ and $\phi_1 = \phi_1(x), \phi_2 = \phi_2(x), \cdots$ are distributions satisfying $\phi_n \to \phi$ as $n \to \infty$, then ϕ is unimodal whenever every ϕ_n is. Finally, a theorem of Hardy and Littlewood concerning "rearrangements" is substantially equivalent to the following fact (cf. [12], p. 47 and pp. 77-78): If ϕ and ψ are two *symmetric* unimodal distributions, then their convolution $\phi * \psi$ is a (symmetric) unimodal distribution.

In [12], the proof of (†) was based on the preceding three facts. The following proof of the generalization (*) of (†) will be an adaptation of the same proof (the first two of the three facts are needed, of course, only in the particular case of symmetry).

As will be shown in Appendix III below, Lévy's enumeration of all L-distributions (in terms of their Fourier-Stieltjes transforms; cf. [6], pp. 192-193, or [22], pp. 145-151) implies that a distribution $\phi(x)$ is a symmetric L-distribution if and only if there exists a sequence of distributions $\phi_1(x), \phi_2(x), \cdots$ satisfying $\phi_n \to \phi$ as $n \to \infty$, where every $\phi_n(x)$ is a convolution of the form

$$\phi_n(x) = \gamma_h(x) * \phi_{q_1}(p_1 x) * \cdots * \phi_{q_n}(p_n x), \tag{5}$$

in which $\gamma_h(x)$ denotes the symmetric normal distribution of a certain standard deviation $h = h_\phi$ $(\geqq 0)$, the n numbers $p_i = p_i(n)$ are positive, the n numbers $q_i = q_i(n)$ are non-negative and, for every $q = q_i$, the distribution $\phi = \phi_q = \phi_q(x)$ is defined by (4) and (3), that is, by

$$\log F_\phi(t) = -q \int_0^t (\sin^2 u)/u \, du; \qquad -\infty < t < \infty. \tag{6}$$

[In view of (1), this implies that $\phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{sgn} x$ if $q = 0$; in what follows, this trivial distribution, which is certainly symmetric and unimodal, will be excluded, that is, q will be assumed to be positive.]

It now follows from the three facts, mentioned above, that the unimodal character of every symmetric L-distribution will be proved if (and only if) it is shown that (6) defines a unimodal distribution ϕ for every fixed $q > 0$. But it is clear from (6), where $F_\phi = F_{\phi_q}$, and from the product rule of the transforms (1) of convolutions, that $\phi_{q+r} = \phi_q * \phi_r$ whenever $q > 0$ and $r > 0$. Hence, the third of the three facts shows that, instead of proving the unimodal character of ϕ_q for every q on the range $0 < q < \infty$, it is sufficient to prove the same for every q on the range $0 < q < q_0$, where q_0 (> 0) can be chosen arbitrarily small.

Accordingly, it can be assumed that $0 < q < 1$. Then $\alpha = q/4$ and $\beta = 1 - q/2$ are positive numbers satisfying $\alpha + \beta + \alpha = 1$. Hence, if $\mu = \mu(x)$, where $\mu(-\infty) = 0$ and $\mu(\infty) = 1$, denotes the step-function having the jumps α and β at $x = \pm 2$ and $x = 0$ respectively, then μ is a distribution. Clearly, $F_\mu(t)$ is identical with the sum of $1 - \frac{1}{2}q$ and $\frac{1}{2}q\cos 2t$. Consequently,

$$F_\mu(t) = 1 - q\sin^2 t. \tag{7}$$

It is now easy to conclude that there belongs to $\phi = \phi_q$ a distribution $\lambda = \lambda_q$ for which

$$tF_\phi(t) = \int_0^t F_\lambda(s)\,ds \tag{8}$$

is an identity in t. In fact, if a prime denotes differentiation with respect to t, then $F_\phi'/F_\phi = -q(\sin t)^2/t$, by (6) and (5). In view of (7), this means that $(tF_\phi)' = F_\phi F_\mu$. Hence (8) is satisfied by the distribution λ which is the convolution of ϕ and μ.

Finally, since (6) is an even function of t, the distribution ϕ is symmetric, and so the assertion, according to which ϕ is a convex distribution, is equivalent to the statement that "the distribution ϕ is unimodal, with $x = 0$ as a mode." But a theorem of Khintchine states that this will be the case precisely if there exists some distribution λ satisfying (8) (in this regard, cf. chap III of Girault's thesis [21], where further references will also be found; concerning Khintchine's proof, cf. the presentation in [22], pp. 157-160, and K. L. Chung's comments on it in [22], pp. 251-253).

The unimodal character of the distribution $\phi = \phi_q$, defined by (6) and

(5), could be verified by an explicit determination (depending on a contour integration) of the Fourier inversion of $F_\phi(t)$. This would avoid an appeal to Khintchine's general criterion. The applicability of the latter is usually not straightforward, and it became straightforward above only because it was permissible to assume that q is sufficiently small.

The applicability of Khintchine's criterion (8) to the case (7) (for *small* q in $\phi = \mu$) seems to be one of the few instances in which the criterion happens to be amenable enough to lead to an affirmative result. In fact, the practical use of the criterion (which is a necesary and sufficient condition) lies perhaps in the opposite direction in most cases. This is illustrated by the proof of the following fact (†bis):

(†bis) *If* $0 < \alpha \leqq 2$, *then*

$$(1 - \alpha \mid t \mid^\alpha) \exp(- \mid t \mid^\alpha) \qquad (-\infty < t < \infty) \tag{9}$$

is the Fourier-Stieltjes transform $F_\lambda(t)$ *of a (symmetric) distribution function* $\lambda = \lambda_\alpha(x)$.

In fact, (8) is equivalent to $(tF_\phi)' = F_\lambda$ or

$$F_\phi(t) + tF_\phi'(t) = F_\lambda(t). \tag{10}$$

On the other hand, the sum on the left of (10) becomes the product (9) if

$$F_\phi(t) = \exp(- \mid t \mid^\alpha). \tag{11}$$

Since the (non-trivial) symmetric stable distributions ϕ are characterized by (11), where $0 < \alpha \leqq 2$, it follows that (†bis) is equivalent to (†), the result quoted at the beginning of this Appendix. (Note that the rule, used at the end of the proof of (*), cannot be applied this time, since the first factor of the product (9), being unbounded as $t \to \infty$, is certainly not an F.)

A similar argument leads to the following curiosity:

If $\Xi(t) = \xi(\frac{1}{2} + it)$, where $\xi(s)$ *is Riemann's entire function, then there exists a* $p = p(x) \geqq 0$ *for which*

$$\Xi(t) + t\Xi'(t) = \int_0^\infty p(x) \cos tx \, dx, \tag{12}$$

where $\Xi' = d\Xi/dt$, *holds for* $-\infty < t < \infty$.

In view of the formulation (10) of Khintchine's criterion. this is equivalent to the result quoted in the footnote to Section 12. (The result of Jensen

and Hurwitz, referred to in that footnote, is the weaker statement that (12) holds for another $p \geqq 0$, say for $p^* = p^*(x)$, if the second term on the left of (12) is omitted.)

The assertion of (†bis) (concerning Fourier *integrals*) is worth restating in terms of Fourier *series*. First, if $f(x)$ is the density of $\phi(x)$ in an absolutely continuous distribution function $\phi(x)$, then

$$\sum_{k=-\infty}^{\infty} f(x+2\pi k) \tag{13}$$

represents (almost everywhere) a periodic function and, according to Poisson's summation formula, the Fourier series (L) of (13) is $(2\pi)^{-1}$ times

$$\sum_{n=-\infty}^{\infty} F_\phi(n) e^{-inx}. \tag{14}$$

But (14) is the Fourier series of a non-negative function,

$$\sum_{n=-\infty}^{\infty} (1 - a\,|\,rn\,|^a) \exp(-|\,rn\,|^a) e^{-inx},$$

and this Fourier series can be written in the form

$$1 + 2\sum_{n=1}^{\infty} (1 + an^a \log q) q^{n^a} \cos nx \tag{15}$$

by placing $r^a = -\log q$. Since $0 < r < \infty$ means that $0 < q < 1$, it is seen that (†bis) contains the following corollary:

If $0 < a \leqq 2$ *and* $0 < q < 1$, *then the function* (15) *is positive for all real* x. This fact is the more remarkable because the coefficient of $n^a (\to \infty)$ in (15) is a negative constant, $a \log q$ (an interpretation of the $\log q$ results if (15) is written in the form

$$\theta_a + a\partial\theta_a/\partial q, \qquad \theta_a = \theta_a(x; q), \tag{15 bis}$$

where θ_a is the function (1) of Appendix I if the x of (15) is replaced by $2\pi x$).

Actually, the italicized corollary of the formulation (†bis) of (†) contains (†bis) itself. This is seen by an application of the rule

$$\lim_{\epsilon \to 0} \sum_{m=1}^{\infty} \epsilon g(m\epsilon) = \int_0^{\infty} g(u)\,du$$

(Euler-Maclaurin), the provisos of which are amply satisfied in the present case (note that $q \to 1$ as $\epsilon \to 0$). On the other hand, if (15) is multiplied

by $\frac{1}{2}$ and if the argument of a complex variable z is denoted by x, then what results is the real part of

$$\frac{1}{2}+\sum_{n=1}^{\infty}(1+an^{a}\log q)q^{n^{a}}z^{n} \tag{16}$$

on the circumference $|z|=1$. Consequently. (†bis) is *equivalent* to the following theorem:

If $0<a\leqq 2$ and $0<q\leqq 1$, then the real part of the power series (16) is positive in the circle $|z|<1$. In other words, the determinant conditions of the Carathéodory-Toeplitz criterion are satisfied by (16) for *every* positive $q<1$ if $0<a\leqq 2$. An algebraic verification of the positivity of the determinants involved, leading to a direct proof of (†bis), appears to be a hopeless task.*

APPENDIX III.

Symmetric *L*-distributions.

1. Let $\phi=\phi(x)$, where $-\infty<x<\infty$, be a distribution (in the sense that not only $d\phi(x)\geqq 0$ holds but also $\phi(-\infty)=0$ and $\phi(\infty)=1$), and let $F_{\phi}(t)$ denote its Fourier-Stieltjes transform,

$$F_{\phi}(t)=\int_{-\infty}^{\infty}e^{itx}d\phi(x)\,;\quad -\infty<t<\infty. \tag{1}$$

Clearly, (1) goes over into its complex conjugate if $\phi(x)$ is replaced by the distribution $1-\phi(-x)$. If the latter is identical with $\phi(x)$, that is, if

$$\phi(x)+\phi(-x)=1 \text{ for } -\infty<x<\infty \tag{2}$$

---

* Corresponding remarks hold if (16) is weakened to

$$\frac{1}{2}+\sum_{n=1}^{\infty}q^{n^{a}}z^{n^{a}} \tag{16 bis}$$

("weakened" in the sense in which the second of the two functions (15 bis) is "weaker" than the first). In fact, a repetition of the preceding deduction shows that (i) the real part of the power series (16 bis) is positive in the circle $|z|<1$ whenever $0\leqq q\leqq 1$ and $0<a\leqq 2$, and that (ii) the truth of (i) is equivalent to the statement Cauchy's symmetric stable distributions actually exist if $a\leqq 2$; that is, to Lévy's result according to which the function (32) of Section 11 will not become negative if $\beta\leqq 2$. If the Carathéodory-Toeplitz criterion is applied to (16 bis), then (ii) leads to a purely algebraic formulation (but not of course to a proof) of Lévy's existence theorem; that is, of his result referred to under (ii).

(when $\phi(x)$ is normalized by $\phi(x) = \frac{1}{2}\{\phi(x+0) + \phi(x-0)\}$), then ϕ is called symmetric.

A fundamental result of Paul Lévy ([27]; cf. also [6], pp. 180-181, or [23], pp. 160-161) enumerates all function $F_\phi(t)$ for which $\phi(x)$ is an "infinitely divisible" distribution. An inspection of his general result shows that, in the particular case (2) (that is, if (1) is real-valued and/or even), the distribution is symmetric if and only if there exists on the closed half-line $0 \leqq u < \infty$ a function $\mu = \mu(u)$ satisfying

$$\int_1^\infty u^{-2}\, d\mu(u) < \infty \quad \text{and} \quad d\mu(u) \geqq 0 \quad \text{for} \quad 0 \leqq u < \infty, \tag{3}$$

and having the property that (1) becomes representable in the form *

$$F_\phi(t) = \exp\{-\int_0^\infty (\sin tu/u)^2 d\mu(u)\}. \tag{4}$$

Thus there is a one-to-one correspondence, $\phi = \phi^\mu$, between all symmetric, infinitely divisible distributions $\phi(x)$ and all functions $\mu(u)$ satisfying (3) if, for the sake of the one-to-one correspondence, μ is normalized by $\mu(u-0) = \mu(u)$ for $0 < u < \infty$. It is understood that the jump

$$\mu(+0) - \mu(0) = c \geqq 0 \tag{5}$$

need not be 0, and that (4) is meant to be

$$F_\phi(t) = \exp\{-ct^2 - \int_{+0}^\infty (\sin tu/u)^2 d\mu(u)\}. \tag{6}$$

Lévy also succeeded in enumerating the distributions ϕ contained in the subclass of the infinitely divisible distributions which consist of L-distributions, the latter being the distributions defined by Khintchine in terms

* The distributions ϕ defined by (4) and (3), that is, by the particular case (2) of Lévy's result, are precisely the distributions rediscovered by J. von Neumann and I. J. Schoenberg [28] in their "metric geometry of Hilbert screws." This observation is made here because [28] as well as the subsequent publications consulted (cf., e.g., [26], pp. 434-438) on either subject (infinite divisibility, Hilbert screws) fail to mention the identity of the "metric" question with the particular case (2) of Lévy's result.

In this regard, the situation is particularly curious in the more recent text of Hille [26], who, *loc. cit.*, not only fails to mention Kolmogoroff's work on Hilbert space or the related corresponding considerations of [28] but it also ignores [25], even though it is precisely Lie's point of view of infinitesimal generators of Lévy's cyclic semi-groups which underlies [25].

of the addition of "asymptotically constant" random variables; cf. [6], pp. 192-193, or [22], pp. 145-151. An inspection of this result of Lévy shows that, in the particular case (2), a distribution $\phi(x)$ is an L-distribution if and only if it is a $\phi(x) = \phi^{\mu}(x)$ in which $\mu = \mu(u)$, besides satisfying (3), has the following properties: $\mu(u)$ is absolutely continuous for $0 < u < \infty$ (while (5) can be positive) and, if $\mu'(u)$ denotes its density (almost everywhere), then (if $\mu'(u)$ is defined on the zero set in an appropriate way) the indefinite integral of $\mu'(u)/u^2$ is a monotone function of $\log u$ for $0 < u < \infty$. It is readily seen that this is equivalent to

$$(7) \qquad d\mu(u) = \mu'(u)du \text{ and } d\{\mu'(u)/u\} \leqq 0, \text{ where } 0 < u < \infty.$$

The object of the following considerations is an analysis of the class of symmetric L-distributions. Except for (i) in Section 2 (a result an extension of which to the unsymmetric case is not available; cf. the footnote at the end of Section 9 below), the symmetry assumption (2) is made only in order to simplify the formulae; the considerations will be such as to apply, *mutatis mutandis*, without the restriction (2) also. Incidentally, if $\phi(x)$ is any L-distribution, then $(1 - \phi(-x)$ is an L-distribution and) the convolution of $\phi(x)$ and $1 - \phi(-x)$ is a symmetric L-distribution; conversely, every L-distribution can be factorized in this manner. This is seen by inspecting the logarithms of the respective transforms (1).

2. In view of Theorem (v) below, a fact proved in Appendix II, implies the following Theorem (i):

(i) *Every symmetric L-distribution ϕ is unimodal.*

By this is meant that $\phi(x)$ is absolutely continuous for $0 < x < \infty$ (hence, by (2), for $-\infty < x < 0$), with a density which is a monotone function of $|x|$ (if $-\infty < x < \infty$ but $x \neq 0$; if $x = 0$, then $\phi(x)$ can have a jump).

(ii) *A distribution ϕ is a symmetric L-distribution if and only if there belong to it a constant $c \geqq 0$ and, on the open half-line $0 < u < \infty$, a function $\lambda = \lambda(u)$ satisfying*

$$(8) \qquad \lambda(u) \geqq 0,\ d\lambda(u) \leqq 0,\ \int_1^{\infty} \lambda(u)/u\, du < \infty,\ \int_{+0}^{1} \lambda(u)du < \infty,$$

and leading to the following representation of the transform (1) of $\phi(x)$ $= \phi_c{}^{\lambda}(u)$:

$$(9) \qquad F_{\phi}(t) = \exp\{-ct^2 + \int_{+0}^{\infty} S(tu)d\lambda(u)\},$$

where $S(t)$ denotes the (even, entire) function

$$S(t) = \int_0^t (\sin v)^2/v\, dv. \tag{10}$$

In fact, if $\lambda(u)$ is defined to be $\mu'(u)/u$, then it is clear that (3) and (7) together are equivalent to (8), and that the relation (4), being identical with (6) by virtue of (5), appears in the form

$$F_\phi(t) = \exp\{-ct^2 - \int_{+0}^{\infty} \lambda(u)(\sin tu)^2/u\, du\}. \tag{11}$$

But it is clear from (10) that $dS(tu)/du = (\sin tu)^2/u$. Hence, a partial integration shows that the integral occurring in (11) is identical with

$$0 - 0 - \int_{+0}^{\infty} S(tu)\, d\lambda(u),$$

since, in view of (8) and (10), the integrated part, $\lambda(u)S(tu)$, tends to 0 whether $u \to \infty$ or $u \to 0$ (while t is fixed). Since this means that (11) is identical with (9), the assertion of (ii) follows.

(iii) *Every symmetric L-distribution ϕ is absolutely continuous for $0 < x < \infty$ (hence, by (2), for $-\infty < x < 0$). In order that ϕ be absolutely continuous for $-\infty < x < \infty$ (that is, in order that the jump $\phi(+0) - \phi(-0)$ be 0), it is necessary and sufficient that the data c, $\lambda(u)$ determining $\phi = \phi_c^\lambda$ satisfy the following alternative: Either $c > 0$ (while $\lambda(u)$ is arbitrary) or $c = 0$ but*

$$\int_{+0}^{\infty} \lambda(u)/u\, du = \infty. \tag{12}$$

This can be concluded from (i) and (ii), as follows: In view of (1) and of the Riemann-Lebesgue lemma, the first assertion of (iii) implies that $F_\phi(t) \to \phi(+0) - \phi(-0)$ as $t \to \infty$. Hence (8) and (9), where $c \geqq 0$, show that $\phi(+0) = \phi(-0)$ if and only if either $c > 0$ (while $\lambda(u)$ is arbitrary) or $c = 0$ but

$$\int_{+0}^{\infty} \lambda(u)(\sin tu)^2/u\, du \to \infty \text{ as } t \to \infty. \tag{13}$$

Since it is readily seen from (8) that (13) is equivalent to (12), this proves (iii).

3. With reference to any positive constant q, let $\phi_q = \phi_q(x)$ denote the symmetric L-distribution ϕ_c^{λ} for which c is 0 and $\lambda(u)$ is q or 0 according as $u \leqq 1$ or $u > 1$. Then (8) and (12) are satisfied, and (9) reduces to

$$F_{\phi_q}(t) = e^{-qS(t)}, \text{ where } S(t) = \int_0^t (\sin u)^2/u \, du, \tag{14}$$

by (10). Hence (ii) and (iii) imply the first of the following three assertions:

(iv) *If q is a positive constant, then the Fourier inversion of the case* (14) *of* (1) *defines a distribution $\phi(x) = \phi_q(x)$ which is absolutely continuous for $-\infty < x < \infty$* (and is symmetric in the sense of (2), since (14) is an even function). *The density $\phi_q'(x) = \phi_q'(-x)$, where $\phi_q' = d\phi_q/dx$, is a monotone function of $|x|$* (which implies that $\phi_q'(\pm\infty) = 0$, and that $\phi_q'(+0) = \phi_q'(-0)$ exists if it is allowed to be ∞; actually, it will not or will be ∞ according as q does not or does exceed a certain critical value q_0, which will be determined in (vi) below). *For varying q, the distributions ϕ_q, where $0 < q < \infty$, form under convolution a semi-group which is cyclic in q:*

$$\phi_{q_1+q_2} = \phi_{q_1} * \phi_{q_2}, \tag{15}$$

and $\phi_q(x)$ tends, as $q \to +0$, to the (*discontinuous*) *distribution*

$$\phi_0(x) = \tfrac{1}{2} + \tfrac{1}{2} \operatorname{sgn} x. \tag{16}$$

The second assertion of (iv) follows from (i) (it is understood that, when the density $\phi_q'(x)$ is claimed to *be* monotone for $0 < x < \infty$, what is actually meant is that $\phi_q'(x)$ *can be chosen* to be monotone for $0 < x < \infty$, since the density of an absolutely continuous distribution function is defined almost everywhere only; a corresponding understanding will hold when $\phi_q'(x)$ will be claimed to be continuous in (vi) below). Finally, since (15) is equivalent to the statement that the transform $F_q(t)$ belonging to $q = q_1 + q_2$ is the product of the transforms belonging to $q = q_1$ and $q = q_2$, and since the transform (1) of (16) is $F_{\phi_0}(t) \equiv 1$, the truth of (15) and of $\phi_{+0} = \phi_0$ is clear from (14).

(v) *A distribution is a symmetric L-distribution if and only if it is a limit, as $n \to \infty$, of distributions of the form*

$$\phi(x) = \gamma_h(x) * \phi_{q_1}(p_1 x) * \cdots * \phi_{q_n}(p_n x), \tag{17}$$

where $\gamma_h(x)$ is the symmetric normal distribution of standard deviation

h ($\geqq 0$), the distribution $\phi_q(x)$ is defined by (14), *and $q_k = q_k(n) \geqq 0$ and $p_k = p_k(n) > 0$, where $k = 1, \cdots, n$, are $2n$ unspecified constants.*

This follows from (ii) and from Lévy's compactness theorem of the transform (1) of arbitrary distributions ϕ. In fact, it is readily seen from (10) that a given function $F_\phi(t)$ of t (where $-\infty < t < \infty$) is representable in the form (9), with some $c \geqq 0$ and with some $\lambda(u)$ satisfying (8), if and only if $F_\phi(t)$ is a limit (a limit which is uniform on *every fixed* finite t-interval) of functions of the form

$$F_\phi(t) = \exp(-ct^2) \prod_{k=1}^{n} \exp\{-q_k S(t/p_k)\}. \tag{18}$$

But it is clear from (14) and from the definition of γ_h (where $h \geqq 0$, and where (16) is meant to be the case $h = 0$ of γ_h) that (18) is equivalent to (17), where $c = c(h) \geqq 0$ according as $h \geqq 0$.

According to (v), the symemtric L-distributions (and only these) can be reduced to the semi-group $(0 < q < \infty)$ of the particular distributions ϕ_q defined in (iv), if these distributions, after the adjunction of the symmetric normal distributions, are extended by the following three operations: changing of the unit of length on the x-axis, the convolution operation (*), and operation of a limit process $(n \to \infty)$. This is the reason why the distributions ϕ_q, defined in (iv), will now be investigated in detail.

4. If a distribution $\phi(x)$ is absolutely continuous for $-\infty < x < \infty$, then the Fourier inversion of (1) is

$$2\pi\phi'(x) = \int_{-\infty}^{\infty} e^{-ixt} F_\phi(t)\, dt, \qquad \left(\int_{-\infty}^{\infty} = \lim_{T\to\infty} \int_{-T}^{T}\right), \tag{19}$$

valid at all those points x (if any) at which the density $\phi' = d\phi/dx$ satisfies a certain local condition. Such a condition is the monotony of ϕ' in some neighborhood of x (if $\phi'(x)$ is meant to be $\frac{1}{2}\phi'(x+0) + \phi'(x-0)$). It follows therefore from the first and from the second of the assertions of (iv) that (with the preceding parenthetical proviso) the representation (19) of the density of $\phi(x)$ is valid for every $\phi_q(x)$ at every $x \neq 0$. In view of (14), this means that

$$\pi\phi_q'(x) = \int_0^{\infty} e^{-qS(t)} \cos xt\, dt, \text{ where } S(t) = \int_0^{t} (\sin u)^2/u\, du, \tag{20}$$

whenever

$$0 < q < \infty \tag{21}$$

and

$$0 < x < \infty \tag{22}$$

(and (22) can be replaced by $-\infty < x < \infty$, since

$$\phi_q'(-x) = \phi_q'(x), \tag{23}$$

by (2)). But the point $x = 0$ is excluded in this deduction.

A partial integration of (20) shows that, under the assumptions (21) and (22),

$$\pi\phi_q'(x)x/q = \int_0^\infty e^{-qS(t)}(\sin t)^2/t \sin xt\, dt. \tag{24}$$

Since it is clear from (10) that $0 \leqq e^{-S(t)} \leqq t^{-C}$, where $C > 0$ is independent of $t > 0$, it is clear that, for every fixed $q > 0$, the integral (24) is uniformly convergent for $-\infty < x < \infty$. It follows therefore from (24), and from the fact that (23) is monotone (non-increasing) on (22), that the product $\phi_q'(x)x$ (when defined at $x = 0$ as its limit for $x \to 0$) is continuous for $-\infty < x < \infty$. Hence, if $q > 0$ is arbitrary, $\phi_q'(x)$ is continuous on (22).

5. It also follows that there exists a constant $c = c_q$ satisfying

$$\phi_q'(x) \sim c_q/x \text{ as } x \to 0, \text{ where } c_q \geqq 0 \tag{25}$$

(it is understood that (25) means

$$x\phi_q'(x) \to 0 \text{ as } x \to 0 \tag{26}$$

if $c_q = 0$). It will now be shown that $c_q > 0$ or $c_q = 0$ according as q is or is not less than a certain critical value (which turns out to be the value $q = 2$).

It is seen from (10) that, as $t \to \infty$,

$$S(t) \sim \text{const.} \int_1^t du/u, \text{ where const.} = \pi^{-1} \int_0^\pi \sin^2 u\, du = \tfrac{1}{2},$$

hence $S(t) \sim \frac{1}{2} \log t$, and that this can be refined to the existence of a constant C having the property that

$$S(t) - \tfrac{1}{2} \log t \to C \text{ as } t \to \infty \tag{27}$$

(in fact, the existence of such a constant follows from $\Sigma 1/n^2 < \infty$ in the same way as the existence of the Euler-Mascheroni constant). It is clear from (27) that, for every fixed $q > 0$,

$$e^{-qS(t)} \sim Q/t^{\frac{1}{2}q} \text{ as } t \to \infty, \text{ where } Q = Q(q) = e^{-qC} > 0. \tag{28}$$

If $q > 2$, then it follows from (28) that, as $t \to \infty$, the integrand of (24) is majorized, uniformly for $-\infty < x < \infty$, by a constant multiple of $1/t^{1+\epsilon}$, where $\epsilon = \epsilon(q) > 0$. It follows therefore from (24) and (23) that, if $q > 2$, the density $\phi_q'(x)$ remains continuous at $x = 0$ and the graph of $y = y(x)$, where $y(x) = \phi_q'(x) = y(-x)$ and $-\infty < x < \infty$, will have at $x = 0$ a tangent parallel to the x-axis. If, on the other hand, $0 < q \leqq 2$, then, since (20) and (28) show that $\phi_q'(x)$ behaves, as $x \to \pm 0$, in the same way as

$$\int_{+0}^{\infty} t^{-\frac{1}{2}q} \cos xt \, dt, \tag{29}$$

the density will become infinite at $x = 0$, the order of infinity being that of $\log |x|^{-1}$ or of $|x|^{-p}$, where $p = p(q) > 0$, according as $q = 2$ or $0 < q < 2$. Accordingly, the situation is as follows:

(vi) *If q (> 0) is arbitrary, then the density $\phi_q'(x)$ is monotone and continuous for $0 < x < \infty$ (hence, by (23), for $-\infty < x < 0$). If $q > 2$, then $\phi_q'(x)$ is continuous at $x = 0$ also. If $0 < q \leqq 2$, then $\phi_q'(x) \to \infty$ as $x \to \pm 0$.*

In fact, the first assertion of (vi), that concerning any $q > 0$, was the result of Section 4.

6. The second assertion of (vi) can be generalized as follows:

(vii) *If $q > 2m$, where m is a fixed positive integer, then $\phi_q(x)$ has a continuous $(m-1)$-st derivative for $-\infty < x < \infty$.*

In fact, the $(m-1)$-st derivative of the integral is majorized, uniformly for $-\infty < x < \infty$, by a constant multiple of

$$\int_{0}^{\infty} t^{m-1} e^{-qS(t)} \, dt, \tag{30}$$

and (28) shows that (30) is a convergent integral if $m - 1 - \frac{1}{2}q > -1$, that is, if $q > 2m$.

(viii) *If $0 < x < \infty$ (or $-\infty < x < 0$), then, whenever $q > 0$, there exists a (continuous) n-th derivative $d^n\phi_q(x)/dx^n$ for $n = 1, 2, \cdots$.*

For $n = 1$, this was proved (Section 4) by a partial integration, that leading from (20) to (24). For an arbitrary n, the assertion of (viii)

follows, after n partial integrations, in the same way as it did for $n=1$. Actually, (viii) is not the last word, since every $\phi_q(x)$ turns out to be regular (analytic) at every real $x \neq 0$.

There is a general principle concerning arbitrary distributions ϕ, which, roughly, is to the effect that "properties of smoothness," possessed by $\phi(x)$ for $-\infty < x < \infty$, are not decreased, and have a tendency to be increased, if $\phi(x)$ is replaced by $\phi(x) * \psi(x)$, where $\psi(x)$ is any distribution. If this is applied to (15), what results is that, if $0 < q < r$, then ϕ_q is at least as "smooth" as ϕ_r (and perhaps "smoother"). Clearly, (vi) and (vii) contain precise formulations of this expectation. On the other hand, (viii) is a manifestation of the general principle only insofar as the C^∞-character of $\phi_q(x)$ on (22) is claimed for *every* value q (large *or* small); so that the "smoothing effect" (when ϕ_q is replaced by ϕ_r, where $r > q$) takes hold only at $x = 0$, at the point excluded in (viii). This remark will be essential in what follows.

7. Theorems (iv) and (vi)-(vii) are purely analytical in nature; they could hardly be expected from the rôle which, according to (v), the semi-group (15), defined on (4) by the Fourier inversion of the case (14) of (1), plays in the theory of symmetric L-distibutions. As a matter of fact, all assertions corresponding to (iv) and (vi)-(vii) become false for the semi-group of distributions, say ψ_q, which belong to the class of all symmetric, infinitely divisible distributions in the same way as the distributions ϕ_q, discussed above, belong to the more restricted class of all symmetric L-distributions. This will be shown by a discussion of the distributions ψ_q just indicated.

First, if the class of all symmetric L-distributions $\phi(x)$ is replaced by the larger class of all symmetric, infinitely divisible distributions, then (11) and (8) become replaced by (6) and (3). Hence it is clear that the distributions ϕ_q, defined for every $q > 0$ by (14), must be replaced by the distributions ψ_q which, for every $q > 0$, are defined by

$$F_{\psi_q}(t) = e^{-qR(t)}, \text{ where } R(t) = \int_0^{|t|} (\sin u / u)^2 \, du. \tag{31}$$

In fact, it is clear that (4) reduces to (31) if the arbitrary $\mu(u)$, which is subject only to (3), is chosen as follows: $\mu(u) = 0$ or $\mu(u) = q$ according as $0 \leqq u \leqq q$ or $q < u < \infty$. Correspondingly, it is clear that (v) becomes true

for the extended ϕ-class if ϕ_q is replaced by ψ_q in (17) (incidentally, what corresponds to (12) in the appropriate analogue of (iii) is

$$\int_{+0}^{\infty} u^{-2}\, d\mu(u) = \infty\,; \tag{32}$$

cf. [25], Theorem (III), p. 289, formula (8), p. 286, and Corollary, p. 288). Finally, it is clear from (31) that, corresponding to (15),

$$\psi_{q_1} * \psi_{q_2} = \psi_{q_1+q_2} \tag{33}$$

and that $\psi_q \to \psi_0$ as $q \to 0$, if ψ_0 denotes the distribution which in (16) is denoted by ϕ_0.

There is however a fundamental difference between ϕ_q and ψ_q (for any fixed $q > 0$). In fact, such results as (i) or (iv) depend on (27) and so, in particular, on the circumstance that $S(\infty) = \infty$ in (14). In contrast, $R(\infty) < \infty$ in (31), since

$$R(t) = \int_0^t (\sin u/u)^2\, du \to \tfrac{1}{2}\pi \text{ as } t \to \infty. \tag{34}$$

A first implication of this contrast is that the jump $\phi(+0) - \phi(-0)$, which was 0 for $\phi = \phi_q$, is $e^{-\frac{1}{2}\pi q} > 0$ for $\phi = \psi_q$; in fact, $F_{\psi_q}(t) \to e^{-\frac{1}{2}\pi q}$ as $t \to \infty$, by (31) and (34). This contrast alone would not be serious, since the jump can be disposed of by first subtracting from $\psi_q(x)$ the function $e^{-\frac{1}{2}\pi q}\phi_0(x)$, where $\phi_0(x)$ is defined by (16), and then dividing the resulting monotone function

$$\theta_q(x) = \psi_q(x) - e^{-\frac{1}{2}\pi q}\phi_0(x) \tag{35}$$

by the (positive) constant $\psi_q(\infty) - e^{-\frac{1}{2}\pi q}\phi_0(\infty) = 1 - e^{-\frac{1}{2}\pi q}$. In order to simplify the formulae, this trivial re-normalization will be disregarded, that is, the function (35) (which differs from a distribution in a positive constant factor) will be used as it stands. Since $F_{\phi_0}(t) \equiv 1$, by (1) and (16), it follows from (31), (35) and (1) that

$$F_{\theta_q}(t) = F_{\psi_q}(t) - F_{\psi_q}(\infty), \text{ where } F_{\psi_q}(\infty) = e^{-\frac{1}{2}\pi q}. \tag{36}$$

Clearly, (33) becomes a relation connecting the three functions $\theta_q(x)$ belonging to $q = q_1, q_2, q_1 + q_2$, if ψ_q is substituted from (35) in terms of θ_q. But it turns out that (31), in contrast to (15), fails to induce an "increase in smoothness" (cf. the end of Section 6) when q increases.

8. This can be seen as follows:

According to (34), there exists a positive constant C for which

$$\tfrac{1}{2}\pi - R(t) \sim C/t \text{ as } t \to \infty \qquad (C > 0) \tag{37}$$

(this C is not the same as, and will play a more delicate part than, the C occurring in (27)). What corresponds to (28) is that, for every fixed $q > 0$,

$$F_{\theta_q}(t) \sim q/t \text{ as } t \to \infty, \text{ where } q = Q(q) = Cqe^{-\frac{1}{2}\pi q} > 0, \tag{38}$$

as seen from (31), (36) and (37).

Since (38) and the relation $F_{\theta_q}(-t) = F_{\theta_q}(t)$ imply that the integral (over the entire t-region $-\infty < t < \infty$) of the square of $F_{\theta_q}(t)$ is finite, it follows from Plancherel's theorem that the case $\phi = \theta_q$ of (1) belongs to an absolutely continuous $\phi(x)$ possessing a density $\phi'(x)$ the square of which has a finite integral over $-\infty < x < \infty$. In particular, $\theta_q(x)$ is absolutely continuous for $-\infty < x < \infty$, with a density $\theta_q'(x)$ which, corresponding to (19), is given by

$$\pi\theta_q'(x) = \int_0^\infty F_{\theta_q}(t) \cos xt \, dt, \tag{39}$$

as soon as any of the standard conditions for the validity of Fourier's theorem is assured. But the proof of (viii), which depends only on successive partial integrations, can be readily repeated when (14) is replaced by (31). Hence $\theta_q(x)$ has derivatives of arbitrarily high order for $0 < x < \infty$, and therefore for $-\infty < x < 0$. In particular, $\theta_q'(x)$ is differentiable at every $x \neq 0$, and so (39) is valid whenever (21) and (22) hold.

Finally, it is seen from (39) and (38) that, if q (> 0) is fixed, $\theta_q'(x) \to \infty$ as $x \to 0$, whether q be small or large. It follows therefore from (35) and (16) that the same is true if $\theta_q'(x)$ is replaced by $\psi_q'(x)$. Hence nothing like (vii) can hold if $\phi_q(x)$ is replaced by $\psi_q(x)$.

9. In order to complete the proof of the statements made in Section 7, it remains to be shown that the first part of the first assertion of (vi) breaks down if $\phi_q'(x) = \phi_q'(-x)$ is replaced by $\psi_q'(x) = \psi_q'(-x)$; in other words, that it is *not* true that all the distributions $\psi_q(x) = 1 - \psi_q(-x)$ are unimodal.

According to the analogue of (v), formulated in Section 6, every symmetric, infinitely divisible distribution $\phi(x)$ can be derived from the

particular distributions $\psi_q(x)$, belonging to the range (21), by certain trivial processes. According to [12], pp. 77-79, each of these processes preserves symmetric unimodal character.* Since every $\psi_q(x)$ is symmetric, it follows that, if every $\psi_q(x)$ were unimodal, then every symmetric, infinitely divisible $\phi(x)$ had to be unimodal and so, in particular, such as to possess no jump at any x distinct from $x=0$. But this is disproved by the example of the distribution $\phi(x)$ which is the convolution of $\pi_h(x)$ and $1-\pi_h(x)$ (cf. the end of Section 1), where $\pi_h(x)$ denotes Poisson's distribution of standard deviations h (>0).

APPENDIX IV.

Bell-shaped frequency curves mod 1.

Let the case $a=2$ of (1) be written in the form

$$\theta(q;\phi)=1+2\sum_{m=1}^{\infty}q^{m^2}\cos m\phi. \tag{1}$$

There are two (substantially different) fundamental properties of this θ: what results from the "linear transformations" property of the elliptic modular functions (a result the explicit form of which is

$$\theta(q;\phi)=\sum_{k=-\infty}^{\infty}(\pi/Q^2)^{-\frac{1}{2}}\exp\{-(2\pi k+\phi)^2/(2Q)^2\},\quad 2\pi^2Q^2=-\log q, \tag{2}$$

the case $a=2$ of (5)-(6) in Appendix I), and the existence of an Eulerian factorization (in the sense of Hecke),

$$\theta(q;\phi)=\sum_{n=1}^{\infty}(1-q^{2n})(1+2q^{2n-1}\cos x+q^{4n-2}) \tag{3}$$

(Jacobi; cf. Pólya-Szegö, chap. I, no. 53). In (2), the arbitrary parameter

* The symmetry restriction is here essential, since the "rearrangement" lemma of Hardy and Littlewood, used in [12], pp. 77-79, cannot be generalized to the statement that the convolution of two arbitrary (not necessarily symmetric) unimodal distributions is unimodal. This observation, made by Paul Lévy (cf. *Revue Mathématique de l'Union Interbalkanique*, vol. 2 (1939), p. 22), was recently rediscovered by K. L. Chung (*Comptes Rendus*, vol. 236 (1953), pp. 583-584, and [22], pp. 254-255). Chung's counterexample, which he needs in order to disprove the validity of a proof given by Gnedenko and Kolmogoroff (Lapin; cf. [22], p. 160), is substantially the same as Lévy's. The non-existence of an extension of [12], pp. 78-79, seemed to be obvious when [12] was written.

$Q\ (>0)$ corresponds to the choice of the standard deviation of a symmetric normal distribution, a *non-cyclic* distribution $(-\infty<\phi<\infty)$ the density of which is the 0-th term $(k=0)$ of the series (2).

If $f(\phi)$ is any real-valued (and, for the sake of simplicity, continuous) function of period 2π, let $f(\phi)$ be called *unimodal* $(\mathrm{mod}\,2\pi)$ if, in the cyclic interpretation of the (ϕ,f)-diagram, the graph of $f=f(\phi)$ consists of a single convex and of a single concave arc $(\mathrm{mod}\,2\pi)$. If $f(\phi)$ has a continuous second derivative $f''(\phi)$, this means that either $f(\phi)$ is a constant or $f''(\phi)$ changes sign (at most and/or exactly) *twice* within a period. If $f(\phi)=f(-\phi)$, this will be referred to as the *bell-shaped* character $(\mathrm{mod}\,2\pi)$ of $f(\phi)$.

In this terminology, the second (that is, the stronger) of the two properties, referred to in the last but one paragraph of Appendix I, can be formulated as follows: $f(\phi)=\theta(q;\phi)$ *is bell-shaped for every fixed positive* $q<1$. This was proved in [18] (it seems to be surprising that a proof was not given many years earlier; an explanation may be that the issue was made acute only by the rôle of (1) as the density of the "cyclic normal distribution," that is, by the interepretation of (2) as the reduction $\mathrm{mod}\,2\pi$ of the linear "normal frequency law"). If Hilfssatz I of Pólya [31], p. 126, is applied to the second derivative of (1) with respect to ϕ, then the difficulty in proving the bell-shaped character $(\mathrm{mod}\,2\pi)$ of (1) for a fixed $q\ (>0)$ is seen to be a *monotone* increasing function of $q\ (<1)$.

The proof in [18] is not a pretty one: it starts out with (2) but, since (2) in itself cannot succeed,* the main point in the proof depends on an appeal to Rouché's theorem, which seems to be entirely out of place in this "explicit" context. Subsequently, the result of [18] was therefore verified by Koschmieder along "explicit" lines; cf. [29] and [30], where the literature of earlier attempts will be found. But Koschmieder's proof represents the other extreme, since it involves an impressive cross-section of the elliptic *Formelapparat*.

Since (3) leads, cf. [17], to quite a trivial proof of the weaker statement

* As a matter of fact, (2) and the mere circumstance that $g(x)=\exp(-cx^2)$ is a monotone function of $|x|$ if $c>0$ (or, for that matter, that this $g(x)$ is bell-shaped for $-\infty<x<\infty$) fail to imply even what in the last but one paragraph of Appendix I was referred to as the *weaker* property. For it clearly is not true that if $g(x)>0$ is an unspecified even function which is monotone for $0\leqq x<\infty$ (or, for that matter, which is bell-shaped for $-\infty<x<\infty$) and tends to 0 with a sufficient rapidity as $x\to\infty$, then the function $f(\phi)=\sum g(\phi+2\pi k)=f(\phi+2\pi)$, where $k=\cdots,-1,0,1,\cdots$, must be monotone on the interval $0<\phi<\pi$.

that (1) is monotone for $0 \leqq x \leqq \pi$ (at every fixed positive $q < 1$), it was natural to try a similar approach to the stronger statement, that claiming the bell-shaped character $(\mathrm{mod}\, 2\pi)$ of (1). The result turns out to be unexpected, since it shows that the matter has nothing to do with elliptic modular functions as such. In fact, the following lemma will now be proved:

If $c_1, c_2, \cdots$ is any sequence of numbers satisfying

$$0 \leqq c_n \leqq 1 \text{ and } \Sigma c_n < \infty, \tag{4}$$

then the function

$$f(\phi) = \Pi f_n(\phi), \tag{5}$$

where

$$f_n(\phi) = 1 + 2c_n \cos\phi + c_n^2, \tag{6}$$

is bell-shaped $(\mathrm{mod}\, 2\pi)$.

It is clear from (3) that the result of [18] on (1) follows from this lemma by choosing $c_n = q^{2n-1}$; in fact, the factor $\Pi(1 - q^{2n}) = \text{const.} > 0$ is immaterial. Similarly, (6) shows that (5) can be written in the form

$$f(\phi) = C\,\Pi(1 + a_n \cos\phi), \qquad a_n = 2c_n/(1 + c_n^2),$$

where the factor $C = \Pi(1 + c_n^2)$ is immaterial. Since the conditions (4) remain unaltered if every c_n is replaced by the corresponding a_n, it is seen that the lemma can be formulated as follows:

If $a_1, a_2, \cdots$ is a sequence of numbers satisfying $0 \leqq a_n \leqq 1$ and $\Sigma a_n < \infty$, then the function $\Pi(1 + a_n \cos\phi)$ is bell-shaped $(\mathrm{mod}\, 2\pi)$.

For reasons of continuity, it is sufficient to prove the lemma for the case in which the first two of the three conditions (4) are replaced by $0 < c_n < 1$. Then the functions (6) and (5) are positive throughout. Two differentiations of (6) and (5) show that

$$-\tfrac{1}{2} f''(\phi)/f(\phi) = 2(F_2 - F_1^2)\sin^2\phi + F_1 \cos\phi, \tag{7}$$

if $' = d/d\phi$, $F_1^2 = (F_1)^2$, and

$$F_j = F_j(\phi) = \Sigma (c_n/f_n)^j, \qquad f_n = f_n(\phi) = f_n(-\phi), \tag{8}$$

where $j = 1$ or $j = 2$ (eventually, (8) will be needed for $j = 3$ also.) The simple structure of (7) and (8) will lead to a straightforward proof.

It can be assumed that ϕ is confined to the interval $0 < \phi < \pi$. Then

the assertion, to be proved, is that $f''(\phi)$ does not change sign more than once. It is clear that if $t(\phi)$ denotes the trigometric function $\frac{1}{2}\cos\phi/\sin^2\phi$, then (7) shows that $f''(\phi)$ will vanish when and only when the function

$$F_1 - F_2/F_1 \tag{9}$$

of ϕ becomes equal to $t(\phi)$. But the derivative of $t(\phi)$ is seen to be negative (since $0 < \phi < \pi$), and so $t(\phi)$ is decreasing throughout. Hence the proof will be complete if it is shown that the function (9) of ϕ is increasing throughout.

Differentiation of (5) and (8) shows that $F_1' = 2F_2 \sin\phi$ and $F_2' = 4F_3 \sin\phi$. Hence the derivative of (9) is seen to be of the same sign as

$$(F_1^2F_2 + F_2^2 - 2F_1F_3)\sin\phi.$$

Since $\sin\phi > 0$ (for $0 < \phi < \pi$), it follows that the function (9) is increasing throughout if it is true that

$$2F_1F_3 < F_1^2F_2 + F_2^2. \tag{10}$$

Let ϕ be fixed. Then every sum (8) is of the form

$$F_j = \sum_{n=1}^{\infty} p_n f_n^j, \text{ where } p_n > 0, \tag{11}$$

since $c_n > 0$ and $f_n(\phi) > 0$. But (11) implies that $F_2^2 \leqq F_1F_3$ (Schwarz). Hence (10) is certainly true if $2F_3 < F_1F_2 + F_3$, that is, if

$$F_3 < F_1F_2. \tag{12}$$

Finally, since $\Sigma\Sigma\, p_np_m(f_nf_m)^2 > \Sigma\, p_np_n(f_nf_n)^2$ if every p is positive, (12) is clear from (11).

THE JOHNS HOPKINS UNIVERSITY.

REFERENCES.

[1] H. Bateman, *Higher transcendental functions*, vol. 3, chap. XVIII (pp. 206-227), New York, 1955.

[2] G. H. Hardy, *Ramanujan* (chap. XI, pp. 186-211), Cambridge, 1940.

[3] ——— and J. E. Littlewood, "Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes," *Acta Mathematica*, vol. 41 (1918), pp. 119-196.

[4] P. Hartman and A. Wintner, "On the spherical approach to the normal distribution law," *American Journal of Mathematics*, vol. 62 (1940), pp. 759-779.

[5] I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton, 1955.

[6] P. Lévy, *Théorie de l'addition des variables aléatoires*, Paris, 1937.

[7] E. Lindelöf, *Le calcul des résidues* (chap. V, pp. 108-141), Paris, 1905.

[8] H. Mellin, "Die Dirichletschen Reihen, die zahlentheoretischen Funktionen und die unendlichen Produkte von endlichem Geschlecht," *Acta Societatis Scientiarum Fennicae*, vol. 31 (1903), no. 2.

[9] R. E. A. C. Paley and N. Wiener, *Fourier transforms in the complex domain*, New York, 1934.

[10] H. Pollard, "The representation of $\exp(-x^\lambda)$ as a Laplace integral," *Bulletin of the American Mathematical Society*, vol. 52 (1946), pp. 908-910, and "The completely monotonic character of the Mittag-Leffler function $E_a(-x)$," *ibid.*, vol. 54 (1948), pp. 1115-1116.

[11] C. R. Putnam and A. Wintner, "On the addition of symmetric normal frequency curves," *Mathematicae Notae* (Rosario), vol. 11 (1952), pp. 79-86.

[12] A. Wintner, "On a class of Fourier transforms," *American Journal of Mathematics*, vol. 58 (1936), pp. 45-90.

[13] ———, "The singularities of Cauchy's distributions," *Duke Mathematical Journal*, vol. 8 (1941), pp. 678-681.

[14] ———, "The sum formula of Euler-Maclaurin and the inversions of Fourier and Möbius," *American Journal of Mathematics*, vol. 69 (1947), pp. 685-708.

Appendix I.

[15] P. Lévy, "L'addition des variables aléatoires définies sur une circonference," *Bulletin de la Société Mathématique de France*, vol. 67 (1939), pp. 1-41. Cf. also [23], pp. 186-191.

[16] A. Wintner, "On the stable distribution laws," *American Journal of Mathematics*, vol. 55 (1933), pp. 335-339.

[17] ———, "On the shape of the angular case of Cauchy's distribution curves," *Annals of Mathematical Statistics*, vol. 18 (1947), pp. 589-593.

[18] ———, "The real elliptic ϑ_3-function," *Duke Mathematical Journal*, vol. 15 (1948), pp. 407-411.

[19] ———, "On a decomposition into singularities of theta-functions of fractional index," *American Journal of Mathematics*, vol. 71 (1949), pp. 105-108.

[20] F. Zernike, "Wahrscheinlichkeitsrechnung und mathematische Statistik," *Handbuch der Physik*, vol. 3 (1928), pp. 419-492.

Appendix II.

[21] M. Girault, "Les fonctions charactéristiques et leur transformations," *Publications de l'Institut de Statistique de l'Université de Paris*, vol. 4 (1955), pp. 223-298.

[22] B. V. Gnedenko and A. N. Kolmogorov, *Limit distributions for sums of independent random variables*, Cambridge, Mass., 1954.

[23] P. Lévy, *Processus stochastiques et mouvement Brownien*, Paris, 1948.

[24] A. Wintner, "Sur les distribution symétrique à fonctions caractéristiques convexes," *Publications de l'Institut de Statistiques de l'Université de Paris* (to appear).

Appendix III.

[25] P. Hartman and A. Wintner, "On the infinitesimal generators of integral convolutions," *American Journal of Mathematics*, vol. 64 (1942), pp. 273-298.

[26] E. Hille, *Functional analysis and semi-groups*, New York, 1948.

[27] P. Lévy, "Sur les intégrales dont les éléments sont des variables aléatoires indépendentes," *Annali della R. Scuola Normale Superiore di Pisa*, ser. 2, vol. 3 (1934), pp. 337-366; cf. *ibid.*, vol. 4 (1935), pp. 217-218.

[28] J. von Neumann and I. J. Schoenberg, "Fourier integrals and metric geometry," *Transactions of the American Mathematical Society*, vol. 50 (1941), pp. 226-251.

Appendix IV.

[29] L. Koschmieder, "Die Krümmung des Schaubildes der Jacobischen Etafunktion," *Anzeiger der Oesterreichischen Akademie der Wissenschaften*, 1948, pp. 210-214.

[30] ———, "Die Krümmung des Schaubildes der Jacobischen Thetafunktion," *Revista Hispano-Americana*, ser. 4, vol. 8 (1948), pp. 257-260.

[31] G. Pólya, "Qualitatives über Wärmeausgleich," *Zeitschrift für angewandte Mathematik und Mechanik*, vol. 13 (1933), pp. 125-128.

THE DECOMPOSITION THEOREM FOR V-MANIFOLDS.*[1]

By WALTER L. BAILY, JR.

1. Introduction. The purpose of this paper is to give a more or less self-contained development of the analytical tools employed in the theory of harmonic integrals in such a way that the theorems we obtain may be applied to fairly general situations. More specifically we wish to present in fairly general context a proof of the so-called regularity theorem as well as a proof of the orthogonal decomposition theorem for compact V-manifolds in a form that will apply for C^∞ forms with coefficients in a vector bundle. In this way we desire to complement a paper of Satake [9] by obtaining Hodge's theorem for compact V-manifolds, present an analytical background for recent papers of Nakano [7], and lay the essential analytical foundations for a forthcoming paper of the author. In our proof we lay no claim to originality but simply make certain modifications in a proof presented by Kodaira in a course given in 1952-53 which in turn represented a certain modification of an earlier proof, also by Kodaira [4], which had been based on methods of Hadamard.

The author wishes to express his thanks to K. Kodaira for many helpful suggestions and conversations. He also wishes to thank I. Satake for the communication of the unpublished manuscript of his paper in which the notion of V-manifold is introduced, and in which an analog of de Rham's theorem is proved for V-manifolds.

2. V-manifolds and vector bundles. We recall here briefly the definition of V-manifold due to Satake from whose paper we carry over certain definitions for reference. By a local uniformizing (l. u. s.) $\{U, G, \varphi\}$ for an open subset $\tilde{U}$ of a Hausdroff space $\mathcal{V}$ we mean a collection of the following objects:

U: a connected open neighborhood of the origin in R^n.

G: a finite group of C^∞ transformations of U.

* Received September 23, 1955; revised April 10, 1956.

[1] The author wishes to acknowledge that the final correction of manuscript for this paper was carried out while the author was employed on a project supported by a National Science Foundation Grant.

φ: a continuous map of U onto $\tilde{U}$, such that $\varphi \circ \sigma = \varphi$ for all $\sigma \in G$ and such that the induced map of U/G onto U is a homeomorphism. (Here $\circ$ denotes composition of functions.)

Let $\{U, G, \varphi\}$ and $\{U', G', \varphi'\}$ be l. u. s.'s for $\tilde{U}$ and $\tilde{U}'$ respectively such that $\tilde{U} \subset \tilde{U}'$; by an injection of $\{U, G, \varphi\}$ into $\{U', G', \varphi'\}$ we mean a C^∞ isomorphism λ of U into U' such that for any $\sigma \in G$ there exists $\sigma' \in G'$ satisfying the relations $\lambda \circ \sigma = \sigma' \circ \lambda$ and $\varphi = \varphi' \circ \lambda$. Then a C^∞ V-manifold shall consist of a connected Hausdorff space $\mathcal{V}$ and a family $\mathcal{F}$ of l. u. s.'s for open subsets of $\mathcal{V}$ satisfying the following conditions:

(i) If $\{U, G, \varphi\}$, $\{U', G', \varphi'\} \in \mathcal{F}$ and $\tilde{U} = \varphi(U)$ is contained in $U' = \varphi'(U')$, then there exists an injection of $\{U, G, \varphi\}$ into $\{U', G', \varphi'\}$.

(ii) The open sets $\tilde{U}$, for which there exists a l. u. s. $\{U, G, \varphi\} \in \mathcal{F}$, form a basis of open sets in $\mathcal{V}$.

Real and complex V-manifolds are defined in a similar fashion. Moreover, a V-manifold is orientable if all the injections in (i) are orientation preserving and if for any l. u. s. $\{U, G, \varphi\}$ the action on U of each $\sigma \in G$ is orientation-preserving.

If $\mathcal{O}$ is an open subset of $\mathcal{V}$, we denote by $\mathcal{F}_\mathcal{O}$ the subfamily of $\mathcal{F}$ consisting of those l. u. s. $\{U, G, \varphi\}$ such that $\tilde{U} \subset \mathcal{O}$. By a differential form ϕ on $\mathcal{O}$ we mean a collection of differential form $\{\phi_U\}$, where ϕ_U is a differential form on U invariant under G for $\{U, G, \varphi\} \in \mathcal{F}_\mathcal{O}$, such that if $\lambda: \{U', G', \varphi'\} \to \{U, G, \varphi\}$ is an injection, $\lambda^* \phi_U = \phi_{U'}$ where λ^* is the mapping of differential forms dual to λ.

By a V-bundle B over $\mathcal{V}$ with group Γ and fibre F we mean that there is given for each l. u. s. $\{U, G, \varphi\} \in \mathcal{F}$ a bundle B_U over U with group Γ and fibre F together with an anti-isomorphism h_U of G into a group of bundle maps of B_U onto itself such that if b lies in the fibre over $x \in U$, then $h_U(g)b$ lies in the fibre over $g^{-1}x$ for $g \in G$; and moreover, if λ is an injection, $\lambda: \{U, G, \varphi\} \to \{U', G', \varphi'\}$, then we are given a bundle map $\lambda^*: B_{U'} \mid \lambda(U) \to B_U$ satisfying the requirements that if $g \in G$ and g' is the unique element of G' such that $\lambda \circ g = g' \circ \lambda$, then $h_U(g) \circ \lambda^* = \lambda^* \circ h_{U'}(g')$, and that if $\lambda\{U, G, \varphi\} \to \{U', G', \varphi'\}$ and $\lambda': \{U', G', \varphi'\} \to \{U'', G'', \varphi''\}$ are injections, then $(\lambda \circ \lambda')^* = \lambda'^* \circ \lambda^*$.

If $\mathcal{V}$, Γ, F, and all B_U have a certain type of differentiable structure and if each λ^* and each $h_U(g)$, $g \in G$, is compatible with this type of differentiable structure, we say that the given V-bundle has this type of differentiable structure. In this paper we shall be concerned mainly with V-bundles

for which F is a real or complex vector space and Γ a real or complex linear group. Without loss of generality we may always suppose for a given V-bundle B over $\mathcal{V}$ and for all l. u. s. $\{U, G, \varphi\} \in \mathfrak{F}$ that B_U is the product bundle over U; then if B is a V-bundle with q-dimensional vector space as fibre, $h_U(g)$ is a continuous mapping of U into the manifold of non-singular $q \times q$ matrices, for $g \in G$.

If B is a V-bundle over $\mathcal{V}$ and $\mathcal{O}$ an open subset of $\mathcal{V}$, then by a section ϕ of B over $\mathcal{O}$ we shall mean a collection of functions $\{\phi_U\}$, where ϕ_U is a function from U into B_U for each $\{U, G, \varphi\} \in \mathfrak{F}_{\mathcal{O}}$ satisfying the following conditions:

s(i) If $x \in U$, $\phi_U(x)$ is in the fibre over x and $h_U(g)\phi_U(x) = \phi_U(g^{-1}x)$ for each $g \in G$.

s(ii) If $\lambda : \{U, G, \varphi\} \to \{U', G', \varphi'\}$ is an injection, then

$$\lambda^* \phi_{U'}(\lambda x) = \phi_U(x).$$

We speak of ϕ as being measurable, continuous, C^∞, etc., when all the functions ϕ_U are respectively measurable, continuous, C^∞, etc. Moreover, from this definition it is also obvious what we will mean by a C^∞ or analytic real or complex valued function on $\mathcal{O}$. If, moreover, B is a vector bundle, then by a metric $\mathfrak{a}$ for B we shall mean a collection of functions $\{\mathfrak{a}_U\}$, where for each $\{U, G, \varphi\} \in \mathfrak{F}$ $\mathfrak{a}_U$ is a function which assigns to each $x \in U$ a (symmetric or hermitian) positive definite bilinear form $\mathfrak{a}_U(x)$ in the fibre of B_U over x such that the following conditions are fulfilled:

m(i) If α and β belong to the fibre of B_U over x, then

$$\mathfrak{a}_U(x)(\alpha, \beta) = \mathfrak{a}_U(g^{-1}x)(h_U(g)\alpha, h_U(g)\beta)$$

for $g \in G$.

m(ii) If $\lambda : \{U, G, \varphi\} \to \{U', G', \varphi'\}$ is an injection, then

$$\mathfrak{a}_U(x)(\lambda^*\alpha, \lambda^*\beta) = \mathfrak{a}_{U'}(\lambda x)(\alpha, \beta),$$

α and β being in the fibre of $B_{U'}$ over λx.

m(iii) If ψ and ϕ are continuous sections of B, the function $\mathfrak{a}(\psi, \phi)$ on $\mathcal{V}$ defined by

$$\mathfrak{a}(\psi, \phi)(\varphi(x)) = \mathfrak{a}_U(x)(\psi_U(x), \phi_U(x)), \quad x \in U,$$

is continuous.

If $\mathfrak{a}(\psi, \phi)$ is a C^∞ function on $\mathcal{V}$ whenever ϕ and ψ are of class C^∞, then we say that $\mathfrak{a}$ is of class C^∞.

From now on we assume that $\mathcal{V}$ is paracompact. Let $\{\tilde{U}_i\}$ be a locally finite covering of $\mathcal{V}$ by open sets $\tilde{U}_i$ such that $\{U_i, G_i, \varphi_i\} \in \mathcal{F}$. By a C^∞ partition of unity subordinate to $\{\tilde{U}_i\}$, we mean a collection of C^∞ functions $\{l_i\}$ on $\mathcal{V}$ such that the support $\mathfrak{L}(l_i)$ of l_i is contained in $\tilde{U}_i$ and such that for each $y \in \mathcal{V}$, $\sum_i l_i(y) = 1$. It is easily seen that given a locally finite covering $\{\tilde{U}_i\}$ of $\mathcal{V}$, there exists a C^∞ partition of unity subordinate to $\{\tilde{U}_i\}$. We simply let $\{\tilde{V}_i\}$ be a shrinkage of $\{\tilde{U}_i\}$ and by an obvious averaging process on U_i with respect to G_i obtain for each i a C^∞ function m_i on $\mathcal{V}$ such that $m_i = 1$ on $\tilde{V}_i$ and $m_i = 0$ outside $\tilde{U}_i$; we then put $l_i = m_i/(\sum m_i)$.

We can now show that if B is a V-bundle over $\mathcal{V}$ with a vector space as fibre, then there exists a C^∞ metric for B. Let $\{U_i\}$ be a locally finite covering of $\mathcal{V}$ such that $\{U_i, G_i, \varphi_i\} \in \mathcal{F}$ and such that B_{U_i} is the product bundle, and let $\{l_i\}$ be a C^∞ partition of unity subordinate to $\{U_i\}$. Then $B_{U_i} = U_i \times V$, where V is a q-dimensional vector space. Let $v_1, \cdots, v_q$ be a fixed basis for V and let

$$b_{U_i}(x)\,(\sum c_k v_k, \sum d_k v_k) = \sum c_k \bar{d}_k.$$

If α and β belong to $y \times V$, $y \in U_i$, define

$$b_{U_i}{}^*(y)\,(\alpha, \beta) = l_i(\varphi_i(y))\,(\operatorname{ord} G_i)^{-1} \sum_{g \in G_i} b_{U_i}(g^{-1}y)\,(h_{U_i}(g)\alpha, h_{U_i}(g)\beta).$$

Finally define

$$a_{U_i}(x)\,(\alpha, \beta) = \sum_j b_{U_j}{}^*(x_{ij})\,(\lambda_{ij}\alpha, \lambda_{ij}\beta),\ x \in U_i,$$

where the summation extends over all j such that $\varphi_j^{-1} \circ \varphi_i(x)$ is not empty, where x_{ij} is a point of $\varphi_j^{-1} \circ \varphi_i(x)$, and where λ_{ij} is defined as follows: Let $\varphi_i(x) \in \tilde{U} \subset \tilde{U}_i \cap \tilde{U}_j$, $\{U, G, \varphi\} \in \mathcal{F}$, and let $\lambda_i{}^* : B_{U_i} \mid \lambda_i(U) \to B_U$ and $\lambda_j{}^* : B_{U_j} \mid \lambda_j(U) \to B_U$ be the bundle maps associated with injections λ_i and λ_j; then $\lambda_{ij} = \lambda_j{}^{*-1} \circ \lambda_i{}^*$. From the construction of $b_{U_i}{}^*$ it is clear that a_{U_i} is well-defined, that the collection of functions $\{a_{U_i}\}$ satisfy m(i) and m(ii), and that $a(\psi, \phi)$ is a function of class C^∞ if ψ and ϕ are of class C^∞. Finally a_U is uniquely defined by the requirement m(ii) for all $\tilde{U}$ such that $\{U, G, \varphi\} \in \mathcal{F}$, for the totality of all such $\tilde{U}$ is a base of neighborhoods for $\mathcal{V}$.

If B is a V-bundle over $\mathcal{V}$ with a finite dimensional vector space X as fibre, we can define the dual V-bundle B' of B. The fibre X' of B' is the dual space of X and the maps $h_U(g)'$ and $\lambda^{*\prime}$ for B' are defined to be the inverses of the transposes of $h_U(g)$ and λ^* respectively, the latter being the bundle maps previously defined for B. A C^∞ metric a for B gives rise to a

C^∞ metric $\mathfrak{a}'$ for B' in the following way: If $\{U, G, \varphi\} \in \mathfrak{F}$, then for each $x \in U$, $\mathfrak{a}_U(x)$ is a positive definite bilinear form in the fibre X_x of B_U over x. $\mathfrak{a}_U(x)$ gives rise naturally to a positive definite, bilinear form $\mathfrak{a}'_U(x)$ on the dual space X'_x, the fibre of B'_U over x, such that the matrical form of $\mathfrak{a}'_U(x)$ with respect to a fixed basis in X'_x is the transposed inverse of $\mathfrak{a}_U(x)$ with respect to a dual basis in X_x. It is easily seen that the functions $\mathfrak{a}'_U$ define a C^∞ metric $\mathfrak{a}'$ for B'.

The tangent bundle T of $\mathfrak{V}$ is defined by taking for T_U the tangent bundle of U, for $h_U(g)$ the inverse of the mapping of tangent vectors induced by g, and for λ^* the inverse of the mapping of tangent vectors induced by the injection λ. If $\mathfrak{g}$ is a metric for T, then for each $\{U, G, \varphi\} \in \mathfrak{F}$ $\mathfrak{g}_U$ is a G-invariant Riemannian metric for U. The dual bundle T' of T is the bundle of differential 1-forms for $\mathfrak{V}$ which we henceforth denote by A^1. In general we use A^p to denote the bundle of differential p-forms for $\mathfrak{V}$, where, by definition, $A_U{}^p$ is the bundle of differential p-forms over U, $h_U(g) = g^*$, the mapping of differential forms dual to g, and λ^* is the mapping of differential forms dual to λ. The metric $\mathfrak{g}$ on T gives rise to a metric $\mathfrak{g}'_p$ in a well-known manner.

From now on we assume that the V-manifold $\mathfrak{V}$ is oriented. If $\{U, G, \varphi\}$ is a l. u. s., and ϕ is an n-form defined on an open subset $\mathfrak{O}$ of $\varphi(U)$, we define the integral of ϕ over $\mathfrak{O}$ by

$$\int_{\mathfrak{O}} \phi = (\text{ord } G)^{-1} \int_{\varphi^{-1}(\mathfrak{O})} \phi_U.$$

From this it is obvious how we define the integral of an n-form over any measurable subset of $\mathfrak{V}$. It is easy to show that if η is any differentiable $(n-1)$-form with compact support on $\mathfrak{V}$, then $\int_{\mathfrak{V}} d\eta = 0$. We let $\{l_i\}$ be a C^∞ partition of unity subordinate to a locally finite covering $\{U_i\}$ of $\mathfrak{V}$ such that $\{U_i, G_i, \varphi_i\} \in \mathfrak{F}$. Then

$$\int_{\mathfrak{V}} d\eta = \int_{\mathfrak{V}} d(\sum_i l_i\eta) = \sum_i \int_{\mathfrak{V}} d(l_i\eta).$$

However, since the support of $l_i\eta$ is compact and contained in U_i,

$$\int_{\mathfrak{V}} d(l_i\eta) = (\text{ord } G_i)^{-1} \int_{U_i} d(l_i'\eta) = 0$$

by application of the ordinary Stokes' theorem, where $l_i'(y) = l_i(\varphi_i(y))$, $y \in U_i$.

If in particular n is the real dimension of $\mathcal{V}$, the n-form dV defined by $dV_U = (\mathfrak{g}'_{nU})^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$, where $x^1, \cdots, x^n$ are the coordinates in U, gives us a measure for $\mathcal{V}$, which measure we also denote by dV. (Note that $\mathfrak{g}'_{nU}$ is a positive real-valued function; in fact if $\mathfrak{g}$ has the matricial form $(\mathfrak{g}_{ij})$ in the coordinates on U, $\mathfrak{g}'_{nU} = \det(\mathfrak{g}_{ij})^{-1}$.) Then if B is any vector V-bundle over $\mathcal{V}$ with a given metric $\mathfrak{a}$ and ϕ, ψ are measurable sections of B, we define $\| \phi \|^2 = \int_{\mathcal{V}} \mathfrak{a}(\phi, \phi) dV$, $(\phi, \psi) = \int_{\mathcal{V}} \mathfrak{a}(\phi, \psi) dV$.

We now suppose $\mathcal{V}$ to be supplied with a C^∞ Riemannian metrc $\mathfrak{g}$. If $\phi = \{\phi_U\}$ is a differentiable form of degree p on $\mathcal{V}$, we take $d\phi$ to be defined by the collection $\{d\phi_U\}$ of differentiable forms of degree $(p+1)$, where $d\phi_U$ is the ordinary exterior derivative of ϕ_U; $d\phi$ is well-defined since d commutes with the mapping of differential forms dual to a mapping of one differentiable manifold into another. Moreover, for any p-form ϕ on $\mathcal{V}$ we define $*\phi = \{*\phi_U\}$ to be the form of degree $(n-p)$ such that for any p-form ψ,

$$\psi_U \wedge *\phi_U = \mathfrak{g}'_p(\psi, \phi) dV_U,$$

where dV_U is the previously defined n-form representing the invariant measure on U. If ϕ is differentiable, we define $\delta\phi = (-1)^{np+n+1} * d * \phi$. Finally we define $\Delta = d\delta + \delta d$.

If ϕ and ψ are differentiable forms of degrees p and $p+1$ respectively, of which at least one has compact support on $\mathcal{V}$, it follows from the identity

$$d(\phi \wedge *\psi) = d\phi \wedge *\psi - \phi \wedge *\delta\psi$$

that $(d\phi, \psi) = (\phi, \delta\psi)$. Hence if ϕ and ψ are twice differentiable p-forms with compact support, $(\Delta\phi, \psi) = (\phi, \Delta\psi)$.

Suppose now that $\mathcal{V}$ is a complex analytic V-manifold of complex dimension n. For each $\{U, G, \varphi\} \in \mathcal{F}$ the complexified tangent bundle $T_U{}^C$ of U splits into a direct sum $T_U{}^C = \mathcal{T}_U \oplus \bar{\mathcal{T}}_U$, where $\mathcal{T}_U$ is a complex analytic bundle with C^n as fibre and where $\bar{\mathcal{T}}_U$ is the bundle with C^n as fibre and whose coordinate transformations are the complex conjugates of those of $\mathcal{T}_U$. This splitting is preserved under the complex analytic bundle maps $h_U(g)$, $g \in G$, and λ^* of T. Correspondingly, the bundle $A^p{}_U$, $p = 1, \cdots, n$, splits in an invariant way into a direct sum $\bigoplus_{r+s=p} A^{r,s}{}_U$ where $A^{r,s}{}_U$ is the bundle of forms of type (r, s) over U, and the exterior differentiation d also splits into $\partial + \bar{\partial}$ where both ∂ and $\bar{\partial}$ commute with all g^* and λ^*. A Hermitian metric ω on $\mathcal{V}$ is simply a Hermitian metric for $\mathcal{T} = \{\mathcal{T}_U\}$, and gives rise to a metric for T, i.e., a Riemannian metric for $\mathcal{V}$, in a canonical fashion; moreover, there is associated with ω a differential 2-form also denoted by ω,

and if $d\omega = 0$, we say that ω is Kähler. We denote by ω'_p the metric on A^p induced by ω. If $B = \{B_U\}$ is a V-bundle over $\mathfrak{V}$ with the complex plane C as fibre and non-zero complex numbers C^* as group, briefly, a complex line bundle, and if $\mathfrak{a}$ is a Hermitian metric for B, then $\mathfrak{a} \otimes \omega'_p = \{\mathfrak{a}_U \otimes \omega'_{pU}\}$ is a metric for $B \otimes A^p = \{B_U \otimes A^p{}_U\}$ whose bundle maps corresponding to elements g of G, $\{U, G, \varphi\} \in \mathfrak{F}$, and injections λ are the tensor products of those of B and A^p. It is clear that $B \otimes A^p = \bigoplus_{r+s=p} B \otimes A^{r,s}$, the terms of the direct sum having an evident meaning. We denote by $\mathfrak{a} \otimes \omega'_{r,s}$ the metric $\mathfrak{a} \otimes \omega'_p$ restricted to $B \otimes A^{r,s}$. We also denote as before by dV_U the n-form defining the invariant measure on U associated with ω_U. A section of $B \otimes A^{r,s}$ is, by definition, a form of type (r,s) with coefficients in B. By $-B$ we mean the complex line bundle for which the non-zero holomorphic functions representing the coordinate transformations and bundle maps $h_U(g)$ and λ^* in particular coordinate systems are the reciprocals of those of B. Then if ϕ is a form of type (r,s) with coefficients in B, we let $\#\phi$ be the form of type $(n-r, n-s)$ with coefficients in $-B$ such that for every form ψ of type (r,s) with coefficients in B we have

$$\psi_U \wedge \#\phi_U = (\mathfrak{a} \otimes \omega'_{r,s})(\psi, \phi) dV_U$$

in U. If ϕ is differentiable, we take $\bar\partial\phi$ to be defined by the collection $\{\bar\partial\phi_U\}$ of forms of type $(r, s+1)$ with coefficients in B; it is easily seen that $\bar\partial\phi$ is well-defined, and in particular that the collection $\{\bar\partial\phi_U\}$ satisfies the conditions s(i) and s(ii). Moreover, we define $\mathfrak{D}\phi = -\#\bar\partial\#\phi$. Finally we put $\square = \mathfrak{D}\bar\partial + \bar\partial\mathfrak{D}$. It follows from the identity

$$d(\phi_U \wedge \#\psi_U) = \bar\partial\phi_U \wedge \#\psi_U - \phi_U \wedge \#\mathfrak{D}\psi_U$$

that if ϕ and ψ are differentiable forms with coefficients in B having compact support and of types (r,s) and $(r, s+1)$ respectively, then $(\phi, \mathfrak{D}\psi) = (\bar\partial\phi, \psi)$. Therefore, if ϕ and ψ are twice differentiable forms of type (r,s) with coefficients in B, then $(\square\phi, \psi) = (\phi, \square\psi)$.

3. Facts about geodesics. In this Section as in the following two Sections we let U be a small neighborhood of the origin ξ_0 in R^n and let G be a finite group of orientation-preserving linear transformations of U onto itself. Instead of the usual Euclidean metric we suppose given some G-invariant, C^∞ Riemannian metric in U. We let this metric be represented by the positive definite matrix function $(\mathfrak{g}_{ij})$ with respect to the coordinates $x^1, \cdots, x^n$ with center at ξ_0. Then Euler's equations for geodesic lines parametrized by $x^i = x^i(t)$ are

$$(3.1) \qquad d(L^{-1}\sum_k \mathfrak{g}_{jk}x^k{}_t)/dt - \tfrac{1}{2}\sum_{i,k} L^{-1}\partial_j\mathfrak{g}_{ik}x^i{}_t x^k{}_t = 0, \qquad j = 1, \cdots, n,$$

where $_t$ denotes differentiation with respect to t, ∂_j is partial differentiation with respect to x_j, and L is the Lagrangian, $L = \sum_{i,k} g_{ik} x^i{}_t x^k{}_t$. If we let s denote arclength, $s = \int_{t_0}^{t_1} L^{\frac{1}{2}}\, dt$, then the equations (3.1) become

$$(3.2) \qquad (d/ds)\left(\sum_k g_{jk}(dx^k/ds)\right) - \tfrac{1}{2}\sum_{i,k} \partial_j g_{ik}(dx^i/ds)(dx^k/ds) = 0.$$

If we put $y_j = \sum_k g_{jk}(dx^k/ds)$, and $H = \frac{1}{2}\sum_{i,k} g^{ik} y_i y_k$, where $\sum_j g^{ji} g_{jk} = \delta^i{}_k$, we have

$$(3.3) \qquad x^k{}_t = (\partial H/\partial y_k), \qquad k = 1, \cdots, n,$$

while our equations (3.2) become

$$(3.4) \qquad (y_k)_t = -\partial H/\partial x^k.$$

We then have the following easily verified facts: (See Kodaira [4] and de Rham [8], pp. 132-138).[2]

3(i) If $\{\xi^i\}$ and $\{\eta_i\}$, $i = 1, \cdots, n$, are sufficiently small real numbers, there exists one and only one system of solutions $x^i = x^i(t; \xi, \eta)$ and $y_i = y_i(t; \xi, \eta)$ satisfying (3.3) and (3.4) and the initial conditions $x^i(0) = \xi^i$, $y_i(0) = \eta_i$; moreover, these solutions are C^∞ functions of ξ, η, and t.

3(ii) $x^i(t; \xi, \eta) = x^i(c^{-1}t; \xi, c\eta)$ and $y_i(t; \xi, \eta) = c^{-1} y_i(c^{-1}t; \xi, c\eta)$, so that if we put $x^i(\xi, \eta) = x^i(1; \xi, \eta)$ and $y_i(\xi, \eta) = y_i(1; \xi, \eta)$, we have

3(ii)′ $y_i(t; \xi, \eta) = t^{-1} y_i(\xi, t\eta)$ and $x^i(t; \xi, \eta) = x^i(\xi, t\eta)$.

3(iii) Letting $\eta^i = \sum_k g^{ik}\eta_k$ we have

$$x^i(\xi, \eta) = \xi^i + \eta^i + O(\eta^2),$$

or, what amounts to the same thing,

$$\eta^i = x^i - \xi^i + O(\eta^2) = x^i - \xi^i + O[(x - \xi)^2],$$

the terms $O(\ )$ and $O[\ ]$ denoting terms of the second order. The coordinates $\eta^1, \cdots, \eta^n$, having center ξ and valid in a small neighborhood of ξ, are called normal coordinates.

---

[2] The existence and uniqueness of solutions $w^i(t; \xi, \eta)$, $y_i(t; \xi, \eta)$ is assured by Theorem 2, p. 42 of [6]. That the solutions are of class C^∞ in ξ, η, t is assured by Theorem 6, p. 101 of [6]. The formula (ii) follows from (i) by direct calculation; (iii) follows from (ii) by Taylor's theorem with remainder in integral form. The rest of the formulas are a matter of direct calculation.

3(iv) Along a geodesic H is constant.

3(v) Letting $r(x,\xi)$ be the geodesic distance between x and ξ, and putting $P(x,\xi) = r(x,\xi)^2$ we have $P(x,\xi) = \sum \eta^i \eta_i = \sum y^i y_i$.

3(vi) $\partial P(x,\xi)/\partial x^i = 2y_i$.

3(vii) $\sum P_i P^i = 4P$, where $P_i = \partial P(x,\xi)/\partial x^i$ and

$$P^i(x,\xi) = \sum_k \mathfrak{g}^{ik}(x) P_k(x,\xi).$$

3(viii) if f is a differentiable function, the derivative $\partial f/\partial r$ of f along a geodesic is given by

$$2r\partial f/\partial r = \sum_i P^i(\partial f/\partial x^i),$$

r being geodesic distance.

3(ix) $$\sum_{k,m} \mathfrak{g}^{km}(x)\,(\partial^2 P(x,\xi)/\partial x^k \partial x^m) = 2n.$$

3(x) $$P_k(\xi,\xi) = 0.$$

4. C^ν approximation of an elementary solution. We retain the notation of the preceding section and for each $g \in G$ we let $\mathfrak{M}_g = (\mathfrak{M}^I{}_{gJ})$, $I,J = 1,\cdots,N$, be a C^∞ mapping of U into the manifold of (real or complex) non-singular $N \times N$ matrices such that if $x \in U$, $g, g' \in G$, then

$$\mathfrak{M}_{gg'}(x) = \mathfrak{M}_g(g'x)\,\mathfrak{M}_{g'}(x).$$

We let $\mathcal{A} = (\mathcal{A}^{IJ})$ be a fixed C^∞ mapping of U into the manifold of (symmetric or Hermitian) positive definite $N \times N$ matrices such that

$${}^t\mathfrak{M}_g(x)\,\mathcal{A}(gx)\,\overline{\mathfrak{M}_g(x)} = \mathcal{A}(x).$$

In other language, $g \to \mathfrak{M}_g$ defines an isomorphism of G into a group of covariant bundle maps of the bundle $U \times R^N$ (real case) or $U \times C^N$ (complex case) onto itself (by $(x,v) \to (gx, \mathfrak{M}_g(x)v)$), and $\mathcal{A}$ is simply a G-invariant metric for the bundle. Let $\phi = (\phi_1,\cdots,\phi_N)$ be a (real or complex) vector-valued function on U, which we also view as an $N \times 1$ column matrix to simplify notation. We define the dual ϕ^* of ϕ to be the vector-valued function with components $(\phi^1,\cdots,\phi^N)$ defined by $\phi^I(x) = \sum_j \mathcal{A}^{IJ}(x)\overline{\phi_J(x)}$, and if $\psi = (\psi_1,\cdots,\psi_N)$ is another N-dimensional vector-valued function, we define

$$(\phi,\psi)_U = \int_U {}^t\phi \cdot \psi^* dV = \int_U \sum_I \psi^I \phi_I\, dV,$$

where dV is the measure arising from the G-invariant metric and ${}^t\phi$ denotes

the transpose of ϕ, and define $\|\phi\|^2_U = (\phi, \phi)_U$, provided the integrals defining these quantities exist. In general ϕ_g is defined by $\phi_g(x) = \mathfrak{M}_g(x)^{-1}\phi(gx)$. We say that ϕ is G-invariant if $\phi_g(x) = \phi(x)$ for $g \in G$, $x \in U$. Then if ϕ and ψ are vector-valued functions, and if ϕ is G-invariant, we have

$$(4.1) \qquad {}^t\phi(x)\mathcal{A}(x)\overline{\psi(x)} = {}^t\phi(gx)\,{}^t\mathfrak{M}_g(x)^{-1}\mathcal{A}(x)\overline{\psi(x)}$$
$$= {}^t\phi(gx)\mathcal{A}(gx)\overline{\mathfrak{M}_g(x)\psi(x)} = {}^t\phi(gx)\mathcal{A}(gx)\overline{\psi_g(gx)}$$

and hence if ϕ and ψ are square summable in the sense that $\|\phi\|^2_U < +\infty$ and $\|\psi\|^2_U < +\infty$,

$$(\phi, \psi)_U = (\phi, \psi_g)_U, \quad g \in G,$$

since the measure dV is G-invariant.

We then suppose given a strongly elliptic differential operator ∇; which, by definition, means an operator which carries a vector-valued function ϕ of class C^2 into a vector-valued function $\nabla\phi$ defined by

$$(4.2) \qquad (\nabla\phi)_I = -\sum_{l,m} \mathfrak{g}^{lm}\partial_l\partial_m\phi_I + \sum_{m,J} a^{mJ}{}_I\partial_m\phi_J + \sum_J b^J{}_I\phi_J,$$

$a^{mJ}{}_I$ and $b^J{}_I$ being functions of class C^∞. We let A^m and B denote respectively the matrices $(a^{mJ}{}_I)$ and $(b^J{}_I)$. Then (4.2) becomes

$$(4.3) \qquad \nabla\phi = -\sum \mathfrak{g}^{lm}\partial_l\partial_m\phi + \sum_m A^m\partial_m\phi + B\phi.$$

We note here that all our dealing for the time being have only to do with a fixed system of coordinates and that $a^{mJ}{}_I$ and $b^J{}_I$ are not generally invariantly defined. We remark in passing that the Laplacian $\Delta = \delta d + d\delta$ is strongly elliptic on the module of differential forms of degree k. See [4].

In what follows we use $\phi \equiv \psi \bmod \mathfrak{N}$ to mean "ϕ equals ψ almost everywhere," and $\phi \in A \bmod \mathfrak{N}$ means there is a member of the class of functions A to which ϕ is equal almost everywhere. Moreover, if $K(x, \xi)$ is an $N \times N$ matrix function on $U \times U$, $\nabla_x K(x, \xi)$ is the $N \times N$ matrix function whose J-th column is obtained by applying ∇ to the J-th column of $K(\ , \xi)$ for a fixed value of ξ. $\int K(x, \xi)\,dV_x$ means that we integrate each entry of K with respect to x holding ξ fixed.

Our first problem in this Section is to find an $N \times N$ matrix function $\bar{\Gamma}^\nu = \{\Gamma^{\nu J}{}_I\}$ such that:

(a) $\nabla_x\bar{\Gamma}^\nu(x, \xi)$ is a matrix function whose entries are real functions which belong to $C^\nu \bmod \mathfrak{N}$ on $U \times U$.

$$\text{(b)}\quad \tilde{\mathbf{\Gamma}}^{\nu}(x,\xi) = \begin{cases} [(n-2)\omega_n]^{-1}P(x,\xi)^{-\tau}U(x,\xi) + \log P(x,\xi)V(x,\xi), & n \geqq 3, \\ -1/4\pi \log P(x,\xi)U(x,\xi) + V(x,\xi), & n = 2, \qquad x \neq \xi, \end{cases}$$

where $\tau = (n-2)/2$, where ω_n is the "area" of the unit sphere in R^n, U and V are matrix functions of class C^∞ in x and ξ, and $U(\xi,\xi)$ is the identity matrix which we denote by $\mathbf{1}$. To achieve this end we tentatively seek to replace the condition (a) by the stronger condition

$$\text{(a)}'\qquad \nabla_x \tilde{\mathbf{\Gamma}}(x,\xi) = 0$$

and begin by writing $\tilde{\mathbf{\Gamma}} = \{\tilde{\mathbf{\Gamma}}^{\nu}{}_{l}(x,\xi)\}$ so that we have

$$\nabla_x \tilde{\mathbf{\Gamma}} = -\sum \mathfrak{g}^{lm}\partial_l\partial_m \tilde{\mathbf{\Gamma}} + \sum_{m} A_m \cdot \partial_m \tilde{\mathbf{\Gamma}} + B \cdot \tilde{\mathbf{\Gamma}}, \tag{4.4}$$

all terms being $N \times N$ matrices and multiplication being matrix multiplication. We then seek to find U and V in the form

$$\text{(c)}\quad \begin{cases} U = M\{\mathbf{1} + \sum_{\kappa=1}^{\infty} P^{\kappa}U_{\kappa}\}, \\ V = M\{\sum_{\kappa=0}^{\infty} P^{\kappa}V_{\kappa}\}, \qquad P^{\kappa} = [P(x,\xi)]^{\kappa}, \end{cases}$$

where M is a matrix which we will shortly define and where U_κ and V_κ are matrix functions of class C^∞ which we will find by a simple recursion formula.

Let $l(P)$ be any (twice differential) scalar function of $P = P(x,\xi)$ and let W be a (twice differentiable) $N \times N$ matrix function of x and ξ. Then direct computation gives

$$\nabla_x(l(P)\cdot W) = l(P)\nabla_x W + l'(P)\{-\sum \mathfrak{g}^{km}\partial_k\partial_m P + \sum A^k P_k - 4r\frac{\partial}{\partial r}\}W - l''(P)4P\cdot W. \tag{4.5}$$

We set in turn

$$\sum \mathfrak{g}^{km}\partial_k\partial_m P - \sum A^k P_k = 2n - 4\mathfrak{N}(x,\xi), \tag{4.6}$$

$$\mathfrak{N}^*(\xi,\eta) = \mathfrak{N}(x(\xi,\eta),\xi), \tag{4.7}$$

$$L(\xi,\eta,t) = t^{-1}\mathfrak{N}^*(\xi,t\eta), \tag{4.8}$$

and obtain immediately from 3(ix) and 3(x) above, that $\mathfrak{N}(\xi,\xi) = 0$, $\mathfrak{N}^*(\xi,0) = 0$, from which it follows[3] that $L(\xi,\eta,t)$ is a function of t,ξ,η of class C^∞; moreover, (4.5) becomes

$$\nabla_x(l(P)\cdot W) = l(P)\cdot\nabla_x W - l'(P)\{2n - 4\mathfrak{N}(x,\xi) + 4r\frac{\partial}{\partial r}\}W - 4l''(P)\cdot P\cdot W. \tag{4.9}$$

[3] This is a consequence of Taylor's theorem with remainder in integral form.

Now put

$$M(x,\xi) = M(x(\xi,t\eta),\xi) = \mathbf{1} + \int_0^t L(t_1)\,dt_1 + \int_0^t \int_0^{t_1} L(t_2)L(t_1)\,dt_2 dt_1$$

$$+ \cdots + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{m-1}} L(t_m)\cdots L(t_1)\,dt_m \cdots dt_1 + \cdots,$$

where $L(t_i) = L(\xi,\eta,t_i)$. That M is actually a function of x and ξ follows from 3(ii), 3(ii)′, and 3(iii), and that the series converges can be seen in the following way: If $|L|$ denotes the norm of the matrix L (see [4], p. 613) and if $|L(t,\xi,\eta)| < K\cdot r(x,\xi)$ for $0 \leqq t \leqq 1$, $|\xi| < \epsilon$, $|\eta| < \epsilon'$, the $(m+1)$-th term of this series has norm less than $m!^{-1}(K^m r^m)$ in this region. Thus the series converges absolutely in this region, and since the series when differentiated term by term with respect to t also converges uniformly, it follows that $dM/dt = LM = t^{-1}\mathfrak{R}^*(\xi,\eta t)M$, or along a geodesic $r(\partial M/\partial r) = \mathfrak{R}(x,\xi)M$. Therefore, if we set $\nabla^* = M^{-1}\nabla_x M$, (4.8) becomes

$$(4.10)\qquad M^{-1}\nabla_x(l(P)MW) = l\nabla^* W - l'(P)\{2n + 4r\frac{\partial}{\partial r}\}W - 4Pl''(P)W.$$

We refer to equations (b) and (c) and consider 3 cases. If n is odd, we put all $V_\kappa = 0$, for notational convenience put $U_{-1} = 0$, and obtain

$$-M^{-1}\nabla_x\Gamma = \sum_{\kappa=0}^{\infty} P^{\kappa-\tau+1}\{4(\kappa-\tau)[\kappa + r\frac{\partial}{\partial r}]U_\kappa - \nabla^* U_{\kappa-1}\};$$

then

$$(4.11)\qquad 4(\kappa-\tau)[\kappa + r\frac{\partial}{\partial r}]U_\kappa - \nabla^* U_{\kappa-1} = 0$$

is satisfied if we take $U_0 = \mathbf{1}$ and define U_κ for $\kappa > 0$ by

$$U_\kappa(x,\xi) = [4(\kappa-\tau)r(x,\xi)^\kappa]^{-1}\int_0^{r(x,\xi)} s^{\kappa-1}\nabla^* U_{\kappa-1}\,ds,$$

i.e., if $x = x(\xi,\eta)$ and if we put $t = s/r$,

$$U_\kappa(x,\xi) = [4(\kappa-\tau)]^{-1}\int_0^1 t^{\kappa-1}\nabla^*{}_x U_{\kappa-1}(x(\xi,t\eta),\xi)\,dt$$

from which it is evident that all U_κ are of class C^∞. If n is even and > 2, U_κ and V_κ are determined by similar recursion formulae (see [4], p. 614) in such a manner that the partial sums in the corresponding expansion of $M^{-1}\nabla_x\Gamma$ are made to "telescope." Finally, the case $n = 2$ is to be treated separately, and it too gives rise to similar recursion formulae (see [4], p. 615). In each case we choose $U_0 = \mathbf{1}$. Then if n is odd we put

$$\tilde{\mathbf{\Gamma}}^\nu(x,\xi) = P^{-\tau}M\sum_{\kappa=0}^{\nu+\frac{1}{2}(n-1)} P^\kappa U_\kappa,$$

from which we obtain, using (4.11),

$$\nabla_x \tilde{\Gamma}^\nu = MP^\nu \sqrt{P}\, \nabla^*_x U_{\nu+\frac{1}{2}(n-1)}$$

which is of class C^ν. If n is even we put

$$\tilde{\Gamma}^\nu = P^{-\tau} M\{\mathbf{1} + \sum_{\kappa=1}^{\nu+1+\tau} P^\kappa U_\kappa\} + M \cdot \log P \sum_{\kappa=0}^{\nu+1} P^\kappa V_\kappa$$

which gives

$$\nabla_x \tilde{\Gamma}^\nu = M\{P^{\nu+1} \nabla^*_x U_{\tau+\nu+1} + (\log P) \cdot P^{\nu+1} \nabla^*_x V_{\nu+1}\},$$

which is of class C^ν. Thus we have found $\tilde{\Gamma}^\nu$ satisfying (a) and (b). It is at this point that the procedure of [4] is modified to eliminate discussion of the convergence of the infinite series appearing in (c) which is established in [4] only for the analytic case; and it is our desire to discuss the C^∞ case. $\tilde{\Gamma}^\nu$ has been called by Kodaira a C^ν approximation to an elementary solution.

For suitably small $\epsilon > 0$ we let U_1 be a neighborhood of ξ_0 such that if $x \in U_1$ and $r(x,\xi)^2 < 3\epsilon$, then $\xi \in U$. We let ρ be a non-negative, real-valued function on the positive reals such that $0 \leqq \rho(y) \leqq 1$ for all y and

$$\rho(y) = 1 \text{ if } y \leqq \epsilon, \text{ and } \rho(y) = 0 \text{ if } y \geqq 2\epsilon,$$

and then define

$$\Gamma_*^\nu(x,\xi) = \rho(r(x,\xi)^2)\, \tilde{\Gamma}^\nu(x,\xi).$$

From now on we assume that ∇ commutes with $\mathfrak{M}_g$, i.e.,

$$\mathfrak{M}_g(x)\,(\nabla\phi)(x) = (\nabla \phi_{g^{-1}})(gx),$$

for each $g \in G$ and every (twice differentiable) ϕ.

We then further define

$$\Gamma^\nu(x,\xi) = (\operatorname{ord} G)^{-1} \sum_{g \in G} \mathfrak{M}_g(x)^{-1} \Gamma_*^\nu(gx, g\xi)\, \mathfrak{M}_g(\xi), \quad x \neq \xi,$$

so that $\mathfrak{M}_g(x)\Gamma^\nu(x,\xi) = \Gamma^\nu(gx, g\xi)\mathfrak{M}_g(\xi)$. Then $Q^\nu(x,\xi)$, is defined by $Q^\nu(x,\xi) = \nabla_x \Gamma^\nu(x,\xi)$. It is clear from the construction of Γ^ν that Q^ν is of class C^ν. Since

$$\Gamma_*^\nu(x,\xi) = [(n-2)\omega_n r(x,\xi)^{n-2}]^{-1}\{M(x,\xi) + O(r(x,\xi))\},$$

where $M(\xi,\xi) = \mathbf{1}$, and since the metric and therefore $r(x,\xi)$ are G-invariant, we have

$$\Gamma^\nu(x,\xi) = [(n-2)\omega_n r(x,\xi)^{n-2}]^{-1}\{M^*(x,\xi) + O(r(x,\xi))\}, \tag{4.12}$$

where $M^*(\xi,\xi)$ is the identity. The case $n = 2$ is slightly different in form,

but we will not give it special attention except when necessary. It should be pointed out here that $\Gamma^\nu(x,\xi)$ and $Q^\nu(x,\xi)$ are not symmetrical in x and ξ.

5. The regularity theorem and auxiliary lemmas. Retaining the notation of the previous sections, we define $\mathfrak{H}$ as the module of vector-valued functions ϕ on U such that $\|\phi\|_U < +\infty$ and $\mathfrak{H}[U_1]$ as the module of vector-valued functions ϕ on U_1 such that $\|\phi\|_U < +\infty$, while we let

$$\mathfrak{H}_\nu = \{\phi \mid \phi \in \mathfrak{H}, \phi \in C^\nu\} \quad \text{and} \quad \mathfrak{H}_\nu[U_1] = \{\phi \mid \phi \in \mathfrak{H}[U_1], \phi \in C^\nu\}.$$

Then $\mathfrak{H}$ and $\mathfrak{H}[U_1]$ with the norms $\|\ \|_U$ and $\|\ \|_{U_1}$ give rise to Hilbert spaces which we also denote by $\mathfrak{H}$ and $\mathfrak{H}[U_1]$, but in which we identify two functions which are equal almost everywhere. It is trivial that the subspace of G-invariant functions are closed since the group G acts continuously in $\mathfrak{H}$ and $\mathfrak{H}[U_1]$.

In what follows it would be possible to use certain formulas from [8], pp. 132-159, if we were speaking of differential forms only. Inasmuch as it is more convenient for our purposes to speak of vector-valued functions instead of differential forms, de Rham's formulas would have to be modified slightly to apply here. The point is that we wish to include the case of differential forms with coefficients in a vector bundle, and in order to avoid complicated indicial notation it seemed desirable to us to speak of vector-valued functions. As remarked in [8], p. 35, however, no really essential changes are needed to pass over to the case of vector-valued forms, and the reader who desires to do so may interpret what we say here in the notation of de Rhams's book without great difficulty.

We define the integral operators Γ^ν and Q^ν by

$$\overline{(\Gamma^\nu\phi)(\xi)} = a(\xi)^{-1} \int_U {}^t\Gamma^\nu(x,\xi)\, a(x)\overline{\phi(x)}\, dV_x$$

and

$$\overline{(Q^\nu\phi)(\xi)} = a(\xi)^{-1} \int_U {}^tQ^\nu(x,\xi)\, a(x)\overline{\phi(x)}\, dV_x$$

provided that the integrands are summable.

Using this notation we wish to prove the following theorems and lemmas from which our desired results follow easily:

LEMMA α. *Suppose μ is a measurable real function on the non-negative real axis such that $|\mu(r)| \leqq Cr^{1-n}$, $C > 0$, and such that $\mu(r) = 0$ if $r^2 \geqq 2\epsilon$.*

Let $W(x,\xi) = \{W(x,\xi)^l{}_J\}$ *be a matrix function of class* C^∞ *on* $U \times U_1$. *Suppose* $\phi \in C^\infty$, $\mathfrak{L}(\phi) \subset U_1$, *where* $\mathfrak{L}$ *denotes support. Then* ψ, *defined by*

$$\psi(\xi) = \int_U \mu(r(x,\xi))W(x,\xi)\phi(x)dV_x, \quad \xi \in U,$$

is of class C^∞ *and* $\mathfrak{L}(\psi) \subset U$.

(The most essential part of Lemma α, namely that ψ is of class C^∞, can be obtained under the stronger hypothesis that μ is $0(r^{2-n})$, and this is sufficient for our purposes in view of (4.12). This weaker result is practically the same as part of Lemma 4, p. 138 of [8].)

LEMMA β. *Suppose* β *is* C^∞, $\mathfrak{L}(\beta) \subset U_1$. *Then* $\int_U \Gamma^\nu(x,\xi)\beta(\xi)dV_\xi$ *is of class* C^∞ *in* x.

LEMMA γ. *If* $\phi \in \mathfrak{H}$, $Q^\nu\phi \in \mathfrak{H}_\nu[U_1]$.

LEMMA δ. Q^ν *is a completely continuous mapping of* $\mathfrak{H}$ *into* $\mathfrak{H}_\nu[U_1]$.

LEMMA ϵ. Γ^ν *is a bounded mapping of* $\mathfrak{H}_\infty$ *into* $\mathfrak{H}_\infty[U_1]$.

Before stating two more theorems, we remark that ∇ has an adjoint ∇^* such that for all (twice differentiable) $\phi, \psi \in \mathfrak{H}$, $(\phi, \nabla\psi) = (\nabla^*\phi, \psi)$, provided either ϕ or ψ has compact support contained in U. In fact, ∇^* is defined by

$$\begin{aligned}\nabla^*\phi &= (\mathfrak{g})^{\frac{1}{2}}\mathcal{A}^{-1}[-\sum_{l,m}\partial_l\partial_m(\mathfrak{g}^{lm}(\mathfrak{g})^{\frac{1}{2}}\mathcal{A}\phi) - \sum_m \partial_m(A^m(\mathfrak{g})^{\frac{1}{2}}\mathcal{A}\phi) + (\mathfrak{g})^{\frac{1}{2}}B\mathcal{A}\phi] \\ &= -\sum_{l,m}\mathfrak{g}^{lm}\partial_l\partial_m\phi + \sum_m A^m{}_*\partial_m\phi + B_*\phi,\end{aligned}$$

where $\mathfrak{g} = \det(\mathfrak{g}_{ij})$ is the determinant of the Riemannian metric form in the given system of coordinates. This formula may be verified by repeated integration by parts. The main fact we need is that the sum of the terms involving second partial derivatives of ϕ is $-\sum_{l,m}\mathfrak{g}^{lm}\partial_l\partial_m\phi$, and since we already know that $\Delta^* = \Delta$ and $\square^* = \square$ (see Section 2), we do not stop to verify the above formula here.

THEOREM A_N. *Let* N *have the same meaning as at the beginning of Section* 4. *Let* $\phi \in \mathfrak{H}$ *be such that* $\mathfrak{L}(\phi) \subset U_1$. *Put*

$$\psi(x) = \int_U \Gamma^\nu(x,\xi)\phi(\xi)dV_\xi.$$

Then $\psi \in C^\infty$, $\mathfrak{L}(\psi) \subset U$, and

$$\nabla\psi(x) = \phi(x) + \int_U Q^\nu(x,\xi)\phi(\xi)dV_\xi, \quad x \in U.$$

THEOREM A^*_N. *Let $\eta \in \mathfrak{J}_\infty$ be such that $\mathfrak{L}(\eta) \subset U$. Then*

$$\Gamma^\nu \nabla^* \eta(\xi) = \eta(\xi) + Q^\nu \eta(\xi), \quad \xi \in U_1.$$

The proofs of Lemmas α-ϵ are relatively simple. As for the proof of Theorems A_N and A^*_N, we shall show that if we assume Lemma α, Theorem A_N is equivalent to Theorem A^*_N; we shall then show that Theorem A_0 implies Theorem A_N, and shall finally prove Theorem $A_0{}^*$.

Let U and S be open subsets of Euclidean space and let $F(x,y)$ be a measurable matrix function on $S \times U$. We say that $\int_U F(x,y)\,dV_y$ converges boundedly on S if $\int_U |F(x,y)|\,dV_y < M$ for all $x \in S$. We will need the following preparatory

LEMMA 0. *Let $F(x,y)$ be a matrix function on $S \times U$ and continuously differentiable in x for $x \neq y$ such that $\partial F/\partial x^i$ is absolutely summable over $K \times U$ for any compact 1-dimensional interval K in S, $i = 1, \cdots, n$. Then $\partial[\int_U F(x,y)\,dV_y]/\partial x^i$ exists and equals $\int_U [\partial F(x,y)/\partial x^i]\,dV_y$ gor $x \in S$. In particular, this is true if $\int_U [\partial F(x,y)/\partial x^i]\,dV_y$ converges boundedly on S.*

Proof. Let $a = (a^1, \cdots, a^n) \in S$ and let $x = (a^1, \cdots, x^i, \cdots, a^n) \in S$ be such that the straight line segment $[a,x]$ lies in S. Then for fixed y

$$\int_a^x [\partial F(u,y)/\partial u^i]\,du^i = F(x,y) - F(a,y) \text{ if } y \notin [a,x].$$

Since $[a,x]$ is a set of measure zero,

$$\int_U \int_a^x [\partial F(u,y)/\partial u^i]\,du^i dV_y = \int_U [F(x,y) - F(a,y)]\,dV_y$$

$$= \int_U F(x,y)\,dV_y + \text{const.}$$

Then by Fubini's theorem,

$$\int_a^x \int_U [\partial F(u,y)/\partial u^i]\,dV_y\,du^i = \int_U F(x,y)\,dV_y + \text{const.}$$

Taking the derivative of both sides, we have

$$\int_U [\partial F(x,y)/\partial x^i]\,dV_y = \partial[\int_U F(x,y)\,dV_y]/\partial x^i. \qquad \text{q. e. d.}$$

1. *Proof of Lemma α.* By assuming U suitably small we may assume

that there exist C^∞ functions h^ν_j, $\nu, j = 1, \cdots, n$ in U such that for each $\xi \in U$ we have $g_{jk}(\xi) = \sum_{\nu=1}^{n} h^\nu_j(\xi) h^\nu_k(\xi)$. This is true since $\{g_{jk}\}$ is positive definite and of class C^∞. We then let $\eta^1, \cdots, \eta^n$ be normal geodesic coordinates with center ξ and write $\theta^\nu(x, \xi) = \sum h^\nu_j(\xi)\eta^j(x, \xi)$. Since $\{h^\nu_j(\xi)\}$ is a non-singular C^∞ matrix, $\theta^1, \cdots, \theta^n$ can be used as local coordinates. (Such coordinates are called orthonormal coordinates.) Then

$$r(x, \xi)^2 = \sum_{\nu=1}^{n} (\theta^\nu(x, \xi))^2.$$

Moreover, $x^i = x^i(\xi, \theta)$ is a C^∞ function of ξ and θ. Therefore

$$\psi(\xi) = \int_U \mu(r(x, \xi)) W(x, \phi(x) dV_x$$
$$= \int_U X(\xi, \theta) \mu([\Sigma(\theta^\nu)^2]^{\frac{1}{2}}) d\theta^1 \cdots d\theta^n,$$

where X is a C^∞ matrix function of ξ and θ. Since $\mu(r) = 0$ if $r^2 > 2\epsilon$, and since $|\mu(r)| \leqq K r^{1-n}$

$$\int_{r^2 \leqq 2\epsilon} \mu([\Sigma(\theta^\nu)^2]^{\frac{1}{2}}) d\theta^1 \cdots d\theta^n \leqq \int_{r^2 \leqq 2\epsilon} K r^{1-n} d\theta^1 \cdots d\theta^n < M < +\infty.$$

Therefore, not only does the integral expressing $\psi(\xi)$ converge absolutely and boundedly for $\xi \in U$, since the support of ϕ is contained in U_1, but so also do the integrals $\int_{r^2 \leqq 2\epsilon} Y(\xi, \theta) \mu([\Sigma\ \theta^\nu)^2]^{\frac{1}{2}}) d\theta^1 \cdots d\theta^n$, where $Y(\xi, \theta)$ is any of the mixed partial derivatives of $X(\xi, \theta)$ with respect to $\xi^1, \cdots, \xi^n$. Hence, by using Lemma 0, it is easily seen that integration and differentiation may be interchanged, and therefore ψ possesses continuous derivatives of all orders.

2. *Proof of Lemma β.* This is an immediate corollary to Lemma α since $\Gamma^\nu(x, \xi)$ is a sum of a finite number of terms having the form $\mu(r(x, \xi))W(x, \xi)$ considered in Lemma α.

3. *Proof of Lemma γ.* We only need show that the integral

$$\mathfrak{I}(\xi) = \int_U {}^t Q(x, \xi) \mathcal{A}(x) \overline{\phi(x)} dV_x$$

is differentiable of class C^ν in U_1 provided ϕ is square summable over U. However, $Q^\nu(x, \xi) = 0$ if $r(x, \xi)^2 \geqq 2\epsilon$, while if $\xi \in U_1$ and $r(x, \xi)^2 \leqq 3\epsilon$, then $x \in U$. Hence, if Y is one of the mixed partial derivatives of $Q^\nu(x, \xi)$

with respect to $\xi^1, \cdots, \xi^n$ of order $\leqq \nu$, then since Y is continuous in both variables, the integral

$$\int_U Y(x,\xi)\mathcal{A}(x)\overline{\phi(x)}dV_x$$

converges absolutely, and boundedly, as we see using Schwarz's inequality, for $x \in U_1$. Therefore, integration and differentiation are interchangeable, and therefore $\mathfrak{A} \in C^\nu$. q. e. d.

4. *Proof of Lemma* 8. This lemma follows immediately from the facts that the kernel $Q^\nu(x,\xi)\mathcal{A}(x)$ of Q^ν is uniformly continuous over $(U_1 + 2\epsilon) \times U_1$, where $U_1 + 2\epsilon = \{x \mid \liminf_{\xi \in U_1} r(x,\xi)^2 \leqq 2\epsilon\}$ (in fact, Q^ν is continuous on $U \times U$), and that $Q(x,\xi) = 0$ if $r(x,\xi)^2 \geqq 2\epsilon$. See Banach [2], p. 97.

5. *Proof of Lemma* ϵ. We wish to prove that there exists $K > 0$ (depending on ν) such that if $\phi \in \mathfrak{J}_\infty$, then $\| \Gamma^\nu\phi \|_{U_1} \leqq K \| \phi \|_U$. (See Kodaira [4], p. 624.) If P is a continuous positive definite (symmetric or Hermitian) $N \times N$ matrix function on U, we define

$$M(P(x)) = \underset{\Sigma_I |u_I|=1}{\text{maximum}} (\Sigma P_{IJ}(x) z_I \bar{z}_J), \quad x \in U,$$

and let $M(P)$ be the least upper bound of the numbers $M(P(x))$ for $x \in U_1 + 2\epsilon$. We let ${}^*\Gamma^\nu(x,\xi) = \Gamma^\nu(x,\xi)\mathcal{A}(x)$ and define

$$|\, {}^*\Gamma^\nu(x,\xi)| = \sqrt{\Sigma_{I,J} |\, {}^*\Gamma^\nu(x,\xi)^{IJ}|^2}.$$

Since $|\, {}^*\Gamma^\nu(x,\xi)| \leqq K r^{1-n}$ uniformly on $U_1 + 2\epsilon$ and since $\Gamma^\nu(x,\xi) = 0$ if $r(x,\xi)^2 \geqq 2\epsilon$, the functions

$$f(\xi) = \int_U |\, {}^*\Gamma^\nu(x,\xi)|\, dV_x \text{ and } g(x) = \int_{U_1} |\, {}^*\Gamma^\nu(x,\xi)|\, dV_\xi$$

are respectively bounded on U_1 and U. We let K_1 be a common bound for f in U_1 and g in U. Then by repeated application of Schwarz's inequality and by Fubini's theorem we have:

$$\begin{aligned}
\| \Gamma^\nu\phi \|^2_{U_1} &= \int_{U_1} {}^t(\Gamma^\nu\phi(\xi))\mathcal{A}(\xi)(\overline{\Gamma^\nu\phi(\xi)})dV_\xi \\
&= \int_{U_1} {}^t\Big(\int_U \Gamma^\nu(x,\xi)\mathcal{A}(x)\overline{\phi(x)}dV_x\Big)\mathcal{A}(\xi)^{-1}\Big(\int_U \overline{\Gamma^\nu(x,\xi)\mathcal{A}(x)}\phi(x)dV_x\Big)dV_\xi \\
&\leqq M(\mathcal{A}^{-1}) \int_{U_1} {}^t\Big(\int_U \Gamma^\nu(x,\xi)\mathcal{A}(x)\overline{\phi(x)}dV_x\Big)\Big(\int_U \overline{\Gamma^\nu(x,\xi)\mathcal{A}(x)}\phi(x)dV_x\Big)dV_\xi \\
&\leqq M(\mathcal{A}^{-1}) \int_{U_1} \Big(\int_U |\, {}^*\Gamma^\nu(x,\xi)|\, dV_x \cdot \int_U |\, {}^*\Gamma^\nu(x,\xi)|\, |\phi(x)|^2 dV_x\Big)dV_\xi
\end{aligned}$$

$$\leqq K_1 \cdot M(\mathcal{A}^{-1}) \int_U \Big(\int_{U_1} |\, {}^*\Gamma^\nu(x,\xi)|\, dV_\xi \Big) |\, \phi(x)|^2\, dV_x$$

$$\leqq N K_1^2 (\mathcal{A}^{-1}) \int_U |\, \phi(x)|^2\, dV_x$$

$$\leqq K_1^2 M(\mathcal{A}^{-1}) \frac{1}{m(\mathcal{A})} \int_U {}^t\phi(x)\mathcal{A}(x)\overline{\phi(x)}\, dV_x = K_3 \parallel \phi \parallel_U^2,$$

where $m(\mathcal{A}) = \underset{x \in U+2\epsilon}{\text{g. l. b.}} \left(\underset{\Sigma_I |z_I|^2 = 1}{\text{minimum}} \sum \mathcal{A}(x)^{IJ} z_I \bar{z}_J\right)$, which is our desired result.

6. *Proof that Theorem* A^*_N *is equivalent to Theorem* A_N. Suppose that $K(x,\xi)$ is an absolutely summable (entry by entry) matrix function on $(U_1 + 2\epsilon) \times U_1$ and that $K(x,\xi) = 0$ if $r(x,\xi)^2 \geqq 2\epsilon$. Let $\eta, \phi \in \mathfrak{J}_\infty$ and suppose that $\mathfrak{L}(\phi) \subset U_1$. Then by Fubini,

$$\int_{U_1} {}^t\phi(\xi) \Big(\int_U K(x,\xi)\mathcal{A}(x)\eta(x)\, dV_x \Big) dV_\xi = \int_U {}^t \Big(\int_{U_1} {}^t K(x,\xi)\phi(\xi)\, dV_\xi \Big) \mathcal{A}(x)\eta(x)\, dV_x$$

Hence we have

$$\int_{U_1} {}^t\phi(\xi) \Big(\int_U {}^t\Gamma^\nu(x,\xi)\mathcal{A}(x)\overline{\nabla^*\eta(x)}\, dV_x \Big) dV_\xi$$

$$= \int_U {}^t \Big\{ \nabla_x \int_{U_1} \Gamma^\nu(x,\xi)\phi(\xi)\, dV_\xi \Big\} \mathcal{A}(x)\overline{\eta(x)}\, dV_x,$$

$$\int_{U_1} {}^t\phi(\xi) \Big(\int_U {}^t Q^\nu(x,\xi)\mathcal{A}(x)\overline{\eta(x)}\, dV_x \Big) dV_\xi$$

$$= \int_U {}^t \Big\{ \int_{U_1} Q^\nu(x,\xi)\phi(\xi)\, dV_\xi \Big\} \mathcal{A}(x)\overline{\eta(x)}\, dV_x,$$

and

$$\int_{U_1} {}^t\phi(\xi)(\mathcal{A}(\xi)\overline{\eta(\xi)})\, dV_\xi = \int_U {}^t\{\phi(x)\}\mathcal{A}(x)\overline{\eta(x)}\, dV_x.$$

Therefore

$$\int_{U_1} {}^t\phi(\xi) \Big[\int_U {}^t\Gamma^\nu(x,\xi)\mathcal{A}(x)\overline{\nabla^*\eta(x)}\, dV_x - \int_U {}^t Q^\nu(x,\xi)\mathcal{A}(x)\overline{\eta(x)}\, dV_x - \mathcal{A}(\xi)\overline{\eta(\xi)} \Big] dV$$

$$= \int_U {}^t \Big\{ \nabla_x \int_{U_1} \Gamma^\nu(x,\xi)\phi(\xi)\, dV_\xi - \int_{U_1} Q^\nu(x,\xi)\phi(\xi)\, dV_\xi - \phi(x) \Big\} \mathcal{A}(x)\overline{\eta(x)}\, dV_x.$$

Because of Lemmas α and δ we see that the right hand side of this equation vanishes for all $\eta, \phi \in \mathfrak{J}_\infty$ such that $\mathfrak{L}(\phi) \subset U_1$ if and only if Theorem A_N is true. while the left hand side vanishes for all such η and ϕ if and only if Theorem A^*_N is true. Hence the equivalence of the two theorems is established.

7. *Proof that Theorem* A_0 *implies Theorem* A_N. Let $K(x,\xi)$ be an $N \times N$ matrix function in U which is differentiable in x. Let $\phi \in \mathfrak{J}_x$, $\mathfrak{L}(\phi) \subset U_1$. Suppose that $\int_U [\partial K(x,\xi)/\partial x^k]\phi(\xi)\, dV_\xi$ converges absolutely,

and boundedly for x in any compact subset of U_1. Then from Lemma 0 we obtain

$$\partial[\int_U K(x,\xi)\phi(\xi)dV_\xi]/\partial x^k = \int_U [\partial K(x,\xi)/\partial x^k]\phi(\xi)dV_\xi, \text{ for } x \in U_1.$$

By hypothesis

$$(5.1) \qquad (\nabla\phi)_I = -\sum_{l,m} g^{lm}\partial_l\partial_m\phi_I + \sum_{m,J} a^{mJ}{}_I\partial_m\phi_J + \sum_J b^J{}_I\phi_J$$

and by construction

$$\Gamma^\nu(x,\xi) = \begin{cases} [(n-2)\omega_n r(x,\xi)^{n-2}]^{-1}\{\mathbf{1} + E_n(x,\xi)\} \\ \qquad = [(n-2)\omega_n r(x,\xi)^{n-2}]^{-1}\cdot\mathbf{1} + A_n(x,\xi), \quad n \neq 2. \\ (4\pi)^{-1}\log r(x,\xi)\{\mathbf{1} + E_2(x,\xi)\} \\ \qquad = (4\pi)^{-1}\log r(x,\xi)\cdot\mathbf{1} + A_2(x,\xi), \quad n = 2. \end{cases}$$

The point we wish to make is that

$$|\partial A_n(x,\xi)/\partial x^k| \leqq C_1 r(x,\xi)^{1-n}, \qquad |\partial^2 A_n(x,\xi)/\partial x^k\partial x^j| \leqq C_2 r(x,\xi)^{1-n},$$

and since $\Gamma^\nu(x,\xi)) = 0$ if $r(x,\xi)^2 \geqq 2\epsilon$, we see that $\partial_k A_n(x,\xi)$ and $\partial_k\partial_j A_n(x,\xi)$ are absolutely summable over U with respect to ξ for $x \in U_1 + \epsilon$, and boundedly for $x \in U_1$. Moreover, $|\Gamma^\nu(x,\xi)| \leqq C_3 r(x,\xi)^{1-n}$ and $|\partial_k\Gamma^\nu(x,\xi)| \leqq C_4 r(x,\xi)^{1-n}$. Hence by our previous remarks, the J-th component of

$$\nabla_x[\int_{U_1}\Gamma^\nu(x,\xi)\phi(\xi)dV_\xi] - \int_{U_1}\nabla_x\Gamma^\nu(x,\xi)\phi(\xi)dV_\xi$$

is simply

$$[(n-2)\omega_n]^{-1}\{-\sum g_{ij}\partial_i\partial_j\int_{U_1}\phi_J(\xi)r(x,\xi)^{2-n}dV_\xi + \int_{U_1}\sum g_{ij}\partial_i\partial_j r(x,\xi)^{2-n}\phi_J(\xi)dV_\xi\}$$
$$+\sum_{k,l}\{\partial_k\partial_l\int_{U_1}D(x,\xi)dV_\xi - \int_{U_1}\partial_k\partial_l D(x,\xi)dV_\xi\}$$
$$+\sum_m\{\partial_m\int_{U_1}C(x,\xi)dV_\xi - \int_{U_1}\partial_m C(x,\xi)dV_\xi\}$$

where D and C are such that $\partial_k\partial_l D(x,\xi)$ and $\partial_m C(x,\xi)$ are absolutely summable over U and boundedly for $x \in U_1$—hence, by Lemma 0, the terms in $\{\ \}$'s vanish and we have that the J-th component of

$$(5.2) \qquad \nabla_x[\int_{U_1}\Gamma^\nu(x,\xi)\phi(\xi)dV_\xi] - \int_{U_1}\nabla_x\Gamma^\nu(x,\xi)\phi(\xi)dV_\xi$$

is

$$(5.3) \quad -\sum g_{ij}(x)[\partial_i\partial_j\int_{U_1}\{[(n-2)\omega_n r(x,\xi)^{n-2}]^{-1}\phi(\xi)dV_\xi\}$$
$$-\int_{U_1}\partial_i\partial_j\{[(n-2)\omega_n r(x,\xi)^{n-2}]^{-1}\phi(\xi)\}dV_\xi].$$

Hence it is sufficient to prove that the latter expression is equal to $\phi_J(x)$, and this just amounts to proving Theorem A_0. We remark further that (5.3) is just what the whole expression (5.2) would equal if ∇ were the Laplacian $\Delta = d\delta + \delta d$, $\mathcal{A}$ were the metric induced by the given invariant Riemannian metric, and ϕ_J were the sole component of a C^∞ differentiable form of degree 0. Hence it is sufficient to prove Theorem A_0 when $\nabla = \Delta$. Therefore it is sufficient to prove Theorem A^*_0 if $\nabla = \Delta$. But since $\Delta = d\delta + \delta d$ is self-adjoint, Theorem A^*_0 becomes:

$$(5.4) \qquad \eta(\xi) = \int_U [\Gamma^\nu(x,\xi)(\Delta\eta)(x) - (\Delta_x\Gamma^\nu(x,\xi))\eta(x)]dV_x, \quad \xi \in U_1,$$

because $\mathcal{A}(\xi) = 1$, Δ is real, i.e., $\bar{\Delta} = \Delta$, and $\Gamma^\nu(x,\xi)$ is 1×1. (Though our operators Γ^ν and Q^ν do not coincide with the operators Ω and Q' of de Rham, our equation (5.4) is practically a special case of Lemma 3(I) p. 146 of [8] as far as the terms of significant contribution are concerned, and the calculations involved in the proof of Lemma 3, pp. 146-148 are practically the same as our calculations in proving (5.4).)

8. *Proof of equation* (5.4). If η, Γ are functions (0-forms), $\delta\eta = 0$, $\delta\Gamma = 0$. Hence

$$\Gamma^\nu(x,\xi)\Delta\eta(x) - (\Delta_x\Gamma^\nu(x,\xi))\eta(x) = \Gamma^\nu(x,\xi)\delta d\eta(x) - (\delta d\Gamma^\nu(x,\xi))\eta(x).$$

On the other hand, for 0-forms Γ and η,

$$\begin{aligned} d*(\Gamma d\eta - \eta d\Gamma) &= d(\Gamma * d\eta - \eta * d\Gamma) \\ &= d\Gamma \wedge *d\eta + \Gamma d*d\eta - d\eta \wedge *d\Gamma - \eta d*d\Gamma \\ &= \Gamma d*d\eta - \eta d*d\Gamma = (\eta\delta d\Gamma - \Gamma\delta d\eta). \end{aligned}$$

Hence it is enough to prove that

$$\eta(\xi) = -\int_U d*(\Gamma d\eta - \eta d\Gamma), \quad \xi \in U_1,$$

where $\Gamma = \Gamma^\nu(x,\xi)$, $\eta = \eta(x)$, and d and $*$ are "with respect to the variable x."

We introduce normal coordinates ζ^i with center ξ such that in these coordinates our given metric $\{g_{ij}\}$ satisfies $g_{ij}(\xi) = \delta_{ij}$. Then the geodesic distance from ξ of a point with normal coordinates $\zeta^1, \cdots, \zeta^n$ is $\sum_{i=1}^n (\zeta^i)^2$. For small $\alpha > 0$ we let $\sigma_\alpha = \{\zeta \mid \sum(\zeta^i)^2 \leqq \alpha^2\}$ be the geodesic sphere of radius α with center ξ. Moreover, we let S_α be the surface of σ_α, $S_\alpha = \{\zeta \mid \sum(\zeta^i)^2 = \alpha^2\}$, and let $U_\alpha = U - \sigma_\alpha$. Then since $\mathfrak{L}(\eta) \subset U$ and $\mathfrak{L}(\eta)$ is compact, we may apply Stokes' Theorem to obtain

$$\int_U d*(\Gamma d\eta - \eta d\Gamma) = \lim_{\alpha\to 0} \int_{U_\alpha} d*(\Gamma d\eta - \eta d\Gamma) = \lim_{\alpha\to 0} \int_{U_\alpha} *(\Gamma d\eta - \eta d\Gamma).$$

Since the $(n-1)$-dimensional "area" of S_α is $0(\alpha^{n-1})$, and since $|\Gamma| \leqq C\alpha^{2-n}$ on S_α, $\lim_{\alpha\to 0}\int_{S_\alpha} *(\Gamma d\eta) = 0$. Therefore

$$\int_U d*(\Gamma d\eta - \eta d\Gamma) = \lim_{\alpha\to 0}\int_{S_\alpha} *(\eta d\Gamma).$$

Moreover, $\Gamma^p(x,\xi) = [(n-2)\omega_n]^{-1} r(x,\xi)^{2-n} + A(x,\xi)$, where $r(x,\xi)^{n-2}A(x,\xi)$ approaches zero as $r(x,\xi)$ approaches zero, so that

$$\lim_{\alpha\to 0}\int_{S_\alpha} *\eta d\Gamma = \lim_{\alpha\to 0}[(n-2)\omega_n]^{-1}\int_{S_\alpha}\eta * d(r(x,\xi)^{2-n}).$$

$\eta \in C^\infty$, and therefore on S_α $\eta = \eta(\xi) + O(\alpha)$. Hence

$$\lim_{\alpha\to 0}\int_{S_\alpha} *\eta d\Gamma = \eta(\xi)[(n-2)\omega_n]^{-1}\lim_{\alpha\to 0}\int_{S_\alpha} * d(r(x,\xi)^{2-n}).$$

We re-employ our previous notation, $P(x,\xi) = r(x,\xi)^2$, $\tau = (n-2)/2$ (the treatment is similar if $n=2$). Then

$$\int_{S_\alpha} * d(r(x,\xi)^{2-n}) = \int_{S_\alpha} * dP^{-\tau} = -\int_{S_\alpha} *\tau P^{-\tau-1} dP = -\tau\alpha^{-n}\int_{S_\alpha} * dP,$$

since $P(x,\xi) = \alpha^2$ for $x \in S_\alpha$. However, $P \in C^\infty$, and hence we may apply Stokes' theorem again obtaining

$$\begin{aligned}\int_{S_\alpha} * d(r(x,\xi)^{2-n}) &= -\tau\alpha^{-n}\int_{\sigma_\alpha} d*dP = \tau\alpha^{-n}\int_{\sigma_\alpha} *\delta dP \\ &= \tau\alpha^{-n}\int_{\sigma_\alpha} *\Delta P = \tau\alpha^{-n}\int_{\sigma_\alpha}\Delta P * 1,\end{aligned}$$

where, in the given system of coordinates, $*1 = \mathfrak{g}^{\frac{1}{2}} d\zeta^1 \wedge \cdots \wedge d\zeta^n$ and $\mathfrak{g}^{\frac{1}{2}} = (\det(g_{ij}))^{\frac{1}{2}}$. Referring to formulas 3(ix), 3(x), and (4.9) with $W=1$ and $l(P) = P$, we see that $\Delta P = 2n - 2\mathfrak{N}(x,\xi)$, where $|\mathfrak{N}(x,\xi)| \leqq K\alpha$ for $x \in \sigma_\alpha$, K being a constant independent of α. Therefore

$$\lim_{\alpha\to 0}\tau\alpha^{-n}\int_{\sigma_\alpha}\Delta P * 1 = \lim_{\alpha\to 0} n(n-2)\alpha^{-n}\int_{\sigma_\alpha} *1.$$

Since $\sqrt{\mathfrak{g}(\xi)} = 1$, $\sqrt{\mathfrak{g}(x)} = 1 + O(\alpha)$ for $x \in \sigma_\alpha$, so that

$$\lim_{\alpha\to 0}\alpha^{-n}\int_{\sigma_\alpha} *1 = \lim_{\alpha\to 0}\alpha^{-n}\cdot\frac{\omega_n}{n}\alpha^n(1+O(\alpha)) = \frac{\omega_n}{n}.$$

Therefore

$$\lim_{\alpha\to 0}\int_{S_\alpha} *(\eta d\Gamma) = \eta(\xi)[(n-2)\omega_n]^{-1}\cdot(n-2)n\cdot\frac{\omega_n}{n} = \eta(\xi),$$

which is our desired result.

We can now prove the regularity theorem:

THEOREM W. *Let ∇ be a strongly elliptic operator as above. Let $\phi \in \mathfrak{H}$ and suppose that for all $\psi \in \mathfrak{H}_\infty$ such that the carrier of ψ is compact and contained in U we have*

$$(w) \qquad \int_U {}^t\phi \mathcal{A} \overline{\nabla^* \psi} dV = 0.$$

Then $\phi \in \mathfrak{H}_\infty[U_1] \bmod \mathfrak{N}$ and $\nabla\phi = 0$.

Suppose, moreover, that all $\mathfrak{M}_g$ commute with ∇ and ∇^*. Then if $\phi \in \mathfrak{H}$ is G-invariant and if (w) holds for all G-invariant $\psi \in \mathfrak{H}_\infty$ such that the carrier of ψ is compact and contained in U, we still have $\phi \in \mathfrak{H}_\infty[U_1]$ and $\nabla\phi = 0$.

Proof. Let $\eta \in \mathfrak{H}_\infty[U_1]$ be such that the carrier of η is compact and contained in U_1. Let

$$\psi(x) = \int_U \Gamma^\nu(x, \xi)\eta(\xi) dV_\xi.$$

Then $\mathfrak{L}(\psi)$ is compact and contained in U, and by Lemma β, $\psi \in C^\infty$. Therefore, by Theorem A_N,

$$0 = \int_U {}^t\phi \mathcal{A} \overline{\nabla^* \psi}\, dV = \int_U {}^t\phi(x) \mathcal{A}(x) [\overline{\eta(x)} + \int_{U_1} \overline{Q^\nu(x, \xi)\eta(\xi)} dV_\xi] dV_x,$$

or, since $\mathfrak{L}(\eta)$ is contained in U_1,

$$\int_{U_1} {}^t\phi \mathcal{A} \overline{\eta}\, dV_\xi = -\int_{U_1} (\int_U {}^t\phi(x) \mathcal{A}(x) \overline{Q^\nu(x, \xi)} dV_x) \overline{\eta(\xi)} dV_\xi.$$

This holds for all $\eta \in \mathfrak{H}_\infty[U_1]$ such that $\mathfrak{L}(\eta)$ is compact and contained in U_1. This shows that on U_1

$${}^t\phi(\xi) = -\int_U {}^t\phi(x) \mathcal{A}(x) \overline{Q^\nu(x, \xi)} dV_x, \quad \bmod \mathfrak{N};$$

but it follows from Lemma δ that the right side is of class C^ν; since ν is arbitrary, we conclude that $\phi \in C^\infty \bmod \mathfrak{N}$. Then by the definition of ∇^* it follows immediately that $\nabla\phi = 0$. This concludes the proof of the first part of the theorem.

Now suppose that ϕ is G-invariant and that for all G-invariant $\psi \in \mathfrak{H}_\infty$, $\mathfrak{L}(\psi) \subset U$, we have $\int_U {}^t\phi \mathcal{A} \nabla^* \phi\, dV = 0$. Let $\theta \in \mathfrak{H}_\infty$, $\mathfrak{L}(\theta) \subset U$. Since our chosen Riemannian metric is G-invariant, since our chosen metric for the bundle is G-invariant, and since $\mathfrak{M}_g$ commutes with ∇ and ∇^* for each $g \in G$, we have, because ϕ is G-invariant,

$$\int_U {}^t(\phi)\mathcal{A}(\overline{\nabla^*\theta})dV = \int_U {}^t(\phi_g)\mathcal{A}(\overline{\nabla^*\theta})dV$$
$$= \int_U {}^t\phi\mathcal{A}(\overline{\nabla^*\theta})_g dV = \int_U {}^t\phi\mathcal{A}\overline{\nabla^*(\theta_g)}dV,$$

where θ_g has the same meaning as at the beginning of Section 4. Therefore, if we put $\theta^* = (\operatorname{ord} G)^{-1}\sum_g \theta_g$, θ^* is G-invariant and

$$\int_U {}^t\phi\mathcal{A}\overline{\nabla^*\theta}\,dV = \int_U {}^t\phi\mathcal{A}\overline{\nabla^*\theta^*}\,dV = 0$$

by hypothesis. Therefore, by the first part of the theorem, $\phi \in C^\infty$, and $\nabla\phi = 0$. Q. E. D.

6. The decomposition theorem for compact V-manifolds. Let $\mathcal{V}$ be a compact C^∞ V-manifold of real dimension n, and let $\mathcal{V}$ be supplied with a C^∞ Riemannian metric $\mathfrak{g}$. Let dV denote the measure on $\mathcal{V}$ associated with $\mathfrak{g}$. We let B be a C^∞ V-bundle over $\mathcal{V}$ having a N-dimensional vector space as fibre, and let $\mathfrak{a}$ be a C^∞ metric for B. Finally, denote by M the module of C^∞ sections of B. We return to our previous assumption that for each $\{U, G, \varphi\} \in \mathfrak{F}$, B_U is the product bundle. Then we employ the notation of the preceding section and in a given l. u. s. let $h_U(g)^{-1} = \mathcal{M}_g$, denote $\mathfrak{a}_U$ simply by $\mathcal{A}$, and dV_U by dV. If ∇ is an endomorphism of M, we say that ∇ is strongly elliptic if (4.2) holds in each l. u. s. $\{U, G, \varphi\}$, it being understood, by definition, that ∇ commutes with the action of $\mathcal{M}_g$ on B_U for each $g \in G$. We let ∇ be a strongly elliptic endomorphism of M for the rest of this section. Then ∇ has a strongly elliptic adjoint ∇^* which satisfies $(\phi, \nabla^*\psi) = (\nabla\phi, \psi)$.

We let $\mathfrak{H}_{\mathcal{V}}$ denote the space of square-summable measurable section of B supplied with inner product of Section 1. We let $\mathcal{H}$ and $\mathcal{H}^*$ denote respectively the null spaces of ∇ and ∇^* in $M \subset \mathfrak{H}_{\mathcal{V}}$. We first observe that $\mathcal{H}$ and $\mathcal{H}^*$ are closed in $\mathfrak{H}_{\mathcal{V}}$. This is an immediate consequence of Theorem W. We then let $\mathcal{L}$ and $\mathcal{L}^*$ denote respectively the orthogonal complements of $\mathcal{H}$ and $\mathcal{H}^*$ in $\mathfrak{H}_{\mathcal{V}}$ and define $\mathcal{L}_\infty = \mathcal{L} \cap M$, $\mathcal{L}_\infty{}^* = \mathcal{L}^* \cap M$. Clearly $M = \mathcal{L}_\infty + \mathcal{H} = \mathcal{L}_\infty{}^* + \mathcal{H}^*$. We first prove

LEMMA M. (*Minimum Eigenvalue*). *There exists $c > 0$ such that for all $\phi \in \mathcal{L}_\infty$, $\phi \neq 0$, $\| \nabla\phi \| / \| \phi \| \geqq c$.*

Proof. If the lemma were false, we could find a sequence $\{\phi_m\}$, $\phi_m \in \mathcal{L}_\infty$, such that $\| \phi_m \| = 1$ and $\lim_{m\to\infty} \| \nabla\phi_m \| = 0$. At each $x \in \mathcal{V}$ we can find a l. u. s. $\{U(x), G, \varphi\}$ and $U_1(x) \subset U(x)$ such that $U(x)$ and $U_1(x)$ enjoy respectively the same properties as U and U_1 of Section 5. We let

$U_1(x^1), \cdots, U_1(x^k)$ cover $\mathcal{V}$ and denote $U(x^i)$ and $U_1(x^i)$ respectively U^i and $U_1{}^i$, $i=1, \cdots, k$. Moreover, for U^i and $U_1{}^i$ we let $\Gamma_i{}^\nu$ and $Q_i{}^\nu$ be the integral operators defined previously for any integer $\nu > 0$. Since $\Gamma_i{}^\nu$ is a bounded operator,

$$(6.1) \quad \lim_{m\to\infty} \| \phi_m + Q_i{}^\nu \phi_m \|_{U_1{}^i} = \lim_{m_j\to\infty} \| \Gamma_i{}^\nu \nabla \phi_m \|_{U_1{}^i} \leqq \lim_{m\to\infty} K_i \| \nabla \phi_m \|_{U_1{}^i} = 0, \quad i=1, \cdots, k,$$

and hence $\lim_{m\to\infty} \| \phi_m + Q_i{}^\nu \phi_m \| = 0$, while on the other hand, since $Q_i{}^\nu$ is completely continuous, we can, by a diagonalization process, find a subsequence $\{\phi_{m_j}\}$ of $\{\phi_m\}$ such that for each i, $\lim_{m_j\to\infty} \| Q_i{}^\nu \phi_{m_j} - \mathrm{X}_i \|_{U_1{}^i} = 0$, where $\mathrm{X}_i \in \mathfrak{J}[U_1{}^i]$. Then by (6.1), $\phi_{m_j} \to -\mathrm{X}_i$, on $U_1{}^i$ and we may take X_i to be G^i-invariant since the subspace of G^i-invariant elements of $\mathfrak{J}[U_1{}^i]$ is closed. It is seen that on $U_1{}^i \cap U_1{}^k$, $\mathrm{X}_i \equiv \mathrm{X}_k \bmod \mathfrak{N}$, and hence ϕ defined by $\phi \equiv -\mathrm{X}_i \bmod \mathfrak{N}$ in $U_1{}^i$ is a well-defined measurable section of B, $\lim \| \phi_{m_j} - \phi \| = 0$, and $\| \phi \| = 1$ since $\| \phi_m \| = 1$. On the other hand, since $\lim \| \nabla \phi_m \| = 0$, and $|(\nabla^* \eta, \phi_{m_j})| = |(\nabla \phi_{m_j}, \eta)| \leqq \| \nabla \phi_{m_j} \| \cdot \| \eta \|$ for each $\eta \in M$, it follows that $(\nabla^* \eta, \phi) = 0$ for all $\eta \in M$, and hence, by Theorem W, $\phi \in C^\infty$ and $\nabla \phi = 0$, i.e., $\phi \in \mathcal{H}$. But then $(\phi_m, \phi) = 0$ for each m and therefore $\| \phi \| = 0$, which contradicts $\| \phi \| = 1$. Therefore our lemma is established.

If ∇ is self-adjoint, i.e., $\nabla = \nabla^*$, then $\mathcal{H} = \mathcal{H}^*$ and $\mathcal{L} = \mathcal{L}^*$. We can now prove

THEOREM B. *Assume that ∇ is self-adjoint. Then for each $\beta \in \mathcal{L}_\infty$ there is a unique $\phi \in \mathcal{L}_\infty$ such that $\nabla \phi = \beta$.*

Proof. Since $\mathcal{H}$ is the null-space of $\nabla = \nabla^*$, it is clear that $\nabla \mathcal{L}_\infty \subset \mathcal{L}_\infty$.

Moreover, $\nabla \mathcal{L}_\infty$ is everywhere dense in $\mathcal{L}$. To see this, let $[\nabla \mathcal{L}_\infty]$ be the closure of $\nabla \mathcal{L}_\infty$ in $\mathcal{L}$ and let $\mathcal{M}$ be its orthogonal complement, $\mathcal{L} = [\nabla \mathcal{L}_\infty] + \mathcal{M}$. Let $m \in \mathcal{M}$. Then $m \perp \nabla \mathcal{L}_\infty$. Since $M = \mathcal{L}_\infty + \mathcal{H}$, $\nabla M = \nabla \mathcal{L}_\infty$ and hence $m \perp \nabla M$. Therefore, by Theorem W, m is of class C^∞ and $\nabla m = 0$; thus $m \in \mathcal{H}$. However, $m \in \mathcal{M} \subset \mathcal{L} \perp \mathcal{H}$. Hence $m = 0$. Therefore $[\nabla \mathcal{L}_\infty] = \mathcal{L}$.

Now let $\beta \in \mathcal{L}_\infty \subset \mathcal{L}$. Then there exists a sequence $\{\phi_m\}$, $\phi_m \in \mathcal{L}_\infty$, such that $\nabla \phi_m \to \beta$. Hence $\nabla \phi_m$ is a Cauchy sequence and thus, by Lemma M, $\{\phi_m\}$ is itself a Cauchy sequence with limit $\phi \in \mathcal{L}$. In each $U_1{}^i$,

$$\Gamma_i{}^\nu \nabla \phi_m = \phi_m + Q_i{}^\nu \phi_m.$$

We know by Lemma ϵ that $\| \Gamma_i{}^\nu \nabla \phi_m - \Gamma_i{}^\nu \beta \|_{U_1{}^i} \leqq C_i \| \nabla \phi_m - \beta \|_{U_i}$, while since $Q_i{}^\nu$ is completely continuous it is automatically bounded and therefore

$\| Q_i^\nu \phi_m - Q_i^\nu \phi \| \leqq C_i^* \| \phi_m - \phi \|$. Therefore $\Gamma_i^\nu \beta \equiv \phi + Q_i^\nu \phi \mod \mathfrak{N}$. By Lemma α, $\Gamma_i^\nu \beta$ is of class C^∞ and by Lemma δ, $Q_i^\nu \phi$ is of class C^ν. Since this is true for each positive integer ν and all i, $\phi \in C^\infty \mod \mathfrak{N}$; thus $\phi \in \mathfrak{L} \mod \mathfrak{N}$. Moreover $\nabla \phi \equiv \beta \mod \mathfrak{N}$; for if $\eta \in M$,

$$(\nabla \phi_m, \eta) = (\phi_m, \nabla \eta), \qquad (\nabla \phi_m, \eta) \to (\beta, \eta),$$

and $(\phi_m, \nabla \eta) \to (\phi, \nabla \eta) = (\nabla \phi, \eta)$. Hence $(\beta - \nabla \phi, \eta) = 0$ for all $\eta \in M$ and therefore $\beta - \nabla \phi = 0$.

If $\nabla \phi' = \beta$, $\| \phi - \phi' \| \leqq C \| \nabla \phi - \nabla \phi' \| = 0$ and hence $\phi \equiv \phi' \mod \mathfrak{N}$. Therefore ϕ is unique and our proof is complete.

We have as a corollary

THEOREM D (*Decomposition Theorem*). *If ∇ is self-adjoint,*

$$M = \nabla M \oplus \mathfrak{H}.$$

Proof. By what we have just proved, $\nabla M = \mathfrak{L}_\infty$, and by construction $M = \mathfrak{L}_\infty \oplus \mathfrak{H}$. Hence $M = \nabla M \oplus \mathfrak{H}$. q. e. d.

THEOREM F. *$\mathfrak{H}$ is finite dimensional.*

Proof. It is sufficient to prove that we can extract a Cauchy sequence from every bounded sequence in $\mathfrak{H}$. Let U^i, U_1^i, Γ_i^ν, and Q_i^ν have the same meanings as in the proof of Lemma M. Let $\{\phi_m\}$ be a bounded sequence in $\mathfrak{H}$. We have

$$0 = \Gamma_i^\nu \nabla \phi_m = \phi_m + Q_i^\nu \phi_m, \text{ on } U_1^i,$$

for each i and every ϕ_m. Since $\{\phi_m\}$ is a bounded sequence and Q_i^ν is completely continuous, we can find by diagonalization a subsequence $\{\phi_{m_j}\}$ such that for each i $\{Q_i^\nu \phi_{m_j}\}$ is a Cauchy sequence on U_1^i under the norm $\| \ \|_{U_1^i}$. Since $\| \phi_{m_j} - \phi_{m_k} \| \leqq \sum_i \| \phi_{m_j} - \phi_{m_k} \|_{U_1^i} = \sum_i \| Q_i^\nu \phi_{m_j} - Q_i^\nu \phi_{m_k} \|_{U_1^i}$, $\{\phi_{m_j}\}$ is itself a Cauchy sequence, and our proof is complete.

7. Hodge's and Kodaira's theorems for V-manifolds. Let $\mathfrak{V}$ be a compact V-manifold supplied with a C^∞ Riemannian metric g. Let A^k be the V-bundle of differential forms of degree k over $\mathfrak{V}$ (see Section 1). Then A^k is a C^∞ V-bundle over $\mathfrak{V}$ with a vector space of dimension $\binom{n}{k}$ as fibre, and the Laplacian $\Delta = d\delta + \delta d$ is a self-adjoint, strongly elliptic endomorphism of the module M^k of C^∞ sections of A^k. Denote respectively by $\mathfrak{H}^k$ and Z^k the null-spaces of Δ and d in M^k. Then by Theorem D, $M^k = \Delta M^k + \mathfrak{H}^k$. It is clear that the spaces $d\delta M$ and $\delta d M$ are orthogonal since $d^2 = 0$ and $\delta^2 = 0$. Therefore $M^k = d\delta M^k \oplus \delta d M^k \oplus \mathfrak{H}^k$. Since

$$(\phi, \Delta\phi) = (d\phi, d\phi) + (\delta\phi, \delta\phi),$$

it is clear that if $\phi \in \mathcal{H}^k$, then $d\phi = 0$. Suppose now that $\phi \in M^k$, $\phi = d\delta a + \delta db + h$, $d\phi = 0$; then $d\delta db = 0$, which shows that $\delta db = 0$. Moreover, it is clear that $d\delta M^k = dM^{k-1}$. Hence $\mathcal{H}^k = Z^k/dM^{k-1}$. On the other hand, Satake has shown that $H^k(\mathcal{V}, R) = Z^k/dM^{k-1}$ where $H^k(\mathcal{V}, R)$ is the k-dimensional Čech cohomology group of $\mathcal{V}$ with real coefficients.

THEOREM H (*Hodge's Theorem*). $H^k(\mathcal{V}, R) \cong \mathcal{H}^k$.

Let B be a complex line bundle over $\mathcal{V}$, and let B be supplied with a C^∞ metric a. Denote by $M(B)^{r,s}$ the module of C^∞ sections of $B \otimes A^{r,s}$, and denote by $\mathcal{H}(B)^{r,s}$ and $Z(B)^{r,s}$ respectively the null-spaces of $\Box$ and ∂ in $M(B)^{r,s}$. Then reasoning parallel to that above (see Kodaira, [5]) yields

THEOREM K (*Kodaira's Theorem*). $\mathcal{H}(B)^{r,s} = Z(B)^{r,s}/\bar{\partial}M(B)^{r,s-1}$.

This will lead to a natural generalization of Dolbealut-Kodaira's Theorem (see [5]) for V-manifolds, which we shall prove in a later paper, and which we shall in turn use to prove an imbedding theorem which slightly generalizes that of [1].

PRINCETON UNIVERSITY.

REFERENCES.

[1] W. L. Baily, "On the quotient of an analytic manifold . . .," *Proceedings of the National Academy of Sciences*, vol. 40 (1954), pp. 804-808.

[2] S. Banach, *Theorie des Operations Lineaires*, Warsaw, 1932.

[3] J. Hadamard, *Le Problem de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris, 1932.

[4] K. Kodaira, "Harmonic fields in Riemannian manifolds," *Annals of Mathematics*, vol. 50 (1949), pp. 587-665.

[5] ———, "On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux," *Proceedings of the National Academy of Sciences*, vol. 39 (1953), pp. 865-868.

[6] F. J. Murray and K. S. Miller, *Existence Theorems for Ordinary Differential Equations*, New York, 1954.

[7] S. Nakano, "On complex analytic vector bundles," *Journal of the Mathematical Society of Japan*, vol. 7 (1955), pp. 1-12.

[8] G. deRham, *Varietes Differentiables*, Paris, 1955.

[9] I. Satake, "On a generalization of the notion of Manifolds," *Proceedings of the National Academy of Sciences*, vol. 42 (1956).

RIEMANN METRICS ASSOCIATED WITH CONVEX BODIES AND NORMED SPACES.*

By DETLEF LAUGWITZ and EDGAR R. LORCH.

1. Introduction. It is the aim of this paper to develop a connection between convex bodies in normed spaces (or Minkowski spaces) and Riemann differential geometry.

We shall use the following notations of tensor calculus with the extension to infinite-dimensional spaces as introduced in [3]. Vectors of the Banach space $\mathfrak{B}^1$ will be denoted by x, y, z, and these letters may bear a superscript: x^i, y^k, etc. Let $\mathfrak{B}_1$ denote the adjoint space of $\mathfrak{B}^1$ (space of covariant vectors). We shall denote the elements of $\mathfrak{B}_1$ by $\xi, \eta, \cdots$, or also by these same letters with a subscript. We shall be concerned only with reflexive spaces $\mathfrak{B}^1$, for which the definition of a tensor of higher order is simply the following. Let us consider bounded multilinear functionals defined upon $\mathfrak{B}^1 \times \cdots \times \mathfrak{B}^1$ (n times) and $\mathfrak{B}_1 \times \cdots \times \mathfrak{B}_1$ (m times). Such a functional will be denoted by $t_{k_1 \cdots k_n}{}^{i_1 \cdots i_m}$. The Banach spaces $\mathfrak{B}_n{}^m$ constituted by these (n, m)-multilinear functionals permit a generalization of the classical summation convention: If in an expression the letter i, say, occurs twice, once as subscript of one functional, and once as superscript of another one, the associated linear operation is to be effected. For instance, $t_k{}^i x^k \eta_i$ stands for the numerical value of the bilinear functional t for $x \in \mathfrak{B}^1$, $\eta \in \mathfrak{B}_1$. For a detailed statement the reader is referred to the paper [3].

For finite-dimensional spaces this notation may be considered in the manner of classical tensor calculus after the choice of any special base. But it should be noted that our interpretation permits a direct operation with vectors and tensors and not only with their components with respect to a base. This point of view seems to be useful for the study of vector spaces. There is no longer any need for proofs of invariance of formulas under a change of base.

These conventions apply to Fréchet differentials. For instance, the derivative of a scalar function $f(x)$ is a bounded linear functional and may be written $\partial f(x)/\partial x^i$.

* Received April 26, 1956.

2. The Riemann metric in $\mathfrak{B}^1$ associated with a convex body $\mathfrak{K}$. We consider a bounded convex body $\mathfrak{K}$ in Hilbert space $\mathfrak{B}^1$ containing the origin 0 as an inner point. It is well known that with $\mathfrak{K}$ a norm or Minkowski metric $F(x)$ is associated which is continuous, positively homogeneous of order 1, and convex, and such that $\mathfrak{K}$ coincides with the point set $F(x) \leqq 1$. We shall suppose that $\mathfrak{K}$ is sufficiently smooth, precisely, that $g(x) = \frac{1}{2}F^2(x)$ has Fréchet differentials (for $x \neq 0$) up to the fifth order.

Denoting the Fréchet derivatives of g by subscripts, we have $g_i \in \mathfrak{B}_1$, $g_{ik} \in \mathfrak{B}_2$, $g_{ijk} \in \mathfrak{B}_3$, etc. We shall now impose a last condition on $\mathfrak{K}$:

$$g_{ik}(x)y^iy^k \geqq m \cdot \| y \|^2 \text{ for } y \neq 0 \text{ with } m = m(x) > 0. \tag{1}$$

This condition implies convexity of $\mathfrak{K}$ in the usual sense [6, § 5]. The functional $g_{ik}(x)$ is homogeneous of order zero, and

$$g_{ik}(x)x^ix^k = 2g(x) = F^2(x). \tag{2}$$

From Euler's theorem on homogeneous functions we have

$$g_{ijk}(x)x^k = 0. \tag{3}$$

It has been stated previously [5, p. 107], that g_{ik} is not only the fundamental tensor of the Minkowski metric $F(x)$, but moreover defines a Riemann metric throughout $\mathfrak{B}^1$ (with the exception of the origin 0):

$$ds^2 = g_{ik}(x)\,dx^i dx^k. \tag{4}$$

In the present paper the authors begin the study of this Riemann metric defined by $\mathfrak{K}$.

3. The curvature tensor. We shall denote by $g^{ij}(x)$ the inverse tensor of $g_{ik}(x)$,

$$g^{ij}(x)g_{jk}(x) = \delta_k{}^i, \tag{5}$$

where $\delta_k{}^i$ denotes the identity operator $\delta_k{}^i x^k = x^i$; g^{ij} exists because of (1) and [3, Satz 3.1]. The affine connection of the Riemann metric $g_{ik}(x)$ is represented by

$$\Gamma_{kj}{}^i = \tfrac{1}{2}g^{ir}(g_{rkj} - g_{kjr} + g_{jrk}) = \tfrac{1}{2}g^{ir}g_{rkj} \tag{6}$$

From this we may compute the curvature tensor (definition for the infinite-dimensional case see [3, p. 137]):

$$R^l{}_{ijk} = \tfrac{1}{4}g^{lr}g^{ms}(g_{mij}g_{rsk} - g_{mik}g_{rsj}). \tag{7}$$

This formula follows by the use of $g_{ijm}g^{jk} + g_{ij}\partial g^{jk}/\partial x^m = 0$, which is obtained by differentiation from (5).

3.1. *For two-dimensional spaces $\mathfrak{B}^1$ the curvature tensor (7) of the Riemann metric $g_{ik}(x)$ associated with any body $\mathfrak{k}$ vanishes identically.*

Proof. Let x denote an arbitrary point, $x \neq 0$. We introduce a special base e_1, e_2 by $e_1 = x$ and $g_{ik}(e_1)e_1{}^i e_2{}^k = 0$, $g_{ik}(e_1)e_2{}^i e_2{}^k = 1$. The components of $\bar{g}_{ijk}(e_1)$ with respect to this base are $\bar{g}_{111}(e_1) = g_{ijk}(e_1)e_1{}^i e_1{}^j e_1{}^k$, etc. By Euler's theorem (3) we obtain

$$0 = g_{ijk}(e_1)e_1{}^k = \bar{g}_{ij1}(e_1), \tag{8}$$

or, in other words, $\bar{g}_{ijk}(e_1) = 0$ if at least one subscript equals 1. Now the only interesting component of the two-dimensional curvature tensor $R_{ijkl} = g_{ih}R^h{}_{jkl}$ is R_{1212}; but if $j = 1$ the expression on the right hand side of (7) vanishes because of (8). From the tensor property of R we have $R^i{}_{jkl} = 0$ in every coordinate system.

Nothing shall be said here about the value of the curvature tensor for higher dimensions except the following remark:

3.2. *If the metric g_{ik} is of constant curvature K, then $K = 0$.*

Proof. $R^i{}_{jkl}(x)x^j = 0$ from (3) and (7). By this the curvature scalar vanishes for any two-dimensional surface element passing through the direction x. This proves 3.2.

4. Applications to E. Cartan's theory of Finsler spaces. The object of E. Cartan's theory of Finsler spaces [1] is a fibre space with a n-dimensional manifold as its basis space and the manifold of directions in n-dimensional vector space as its fibre. (Our results shall hold good for the infinite-dimensional Finsler spaces introduced in [4].) We consider here only the local point of view, that is, the tangent space of one fixed point. For this special case, Cartan's metric is $g_{ik}(x) = \partial^2 g(x)/\partial x^i \partial x^k$, and the components of the euclidean connection are

$$C_j{}^i{}_k(x) = \tfrac{1}{2}g^{ir}(x)g_{rjk}(x).$$

(To simplify notations, we write x instead of Cartan's x', which is possible since we consider a fixed point of Finsler space.) Obviously these geometric objects have the same expressions as the corresponding ones, namely g_{ik} and $\Gamma_{jk}{}^i$, in Section 2 of our paper, if we take for $\mathfrak{k}$ the indicatrix of the local

Minkowski metric. A more interesting connection is the relation between the curvature tensor R_{ijkl} of formula (7) and Cartan's tensor S_{ijkl}

$$R = 2g \cdot S, \tag{9}$$

which follows directly from the analytic expressions (7) and [1, (XVI), p. 34]. From (9), (1), and our Theorem 3.2 follows at once:

4.1. *For two-dimensional Finsler spaces Cartan's tensor S_{ijkl} vanishes identically.*

The Landsberg angle $d\phi$ ([2]; métrique angulaire en un point, [1, p. 13]) of two directions x, $x+dx$ ($F(x)=F(x+dx)=1$) is defined by $d\phi^2 = g_{ik}(x)dx^i dx^k$. Its geometrical meaning in the Riemann geometry introduced here is: *$d\phi$ equals the Riemann length of the arc joining the end points of the two unit vectors.* (See also O. Varga [8])

5. Connections with results of Varga. Recently O. Varga [8] has studied other Riemann geometries associated with $\mathfrak{C}$. In our terminology we may state his point of view simply as follows. Varga introduces the tensor $\gamma_{\alpha\beta} = x_\alpha{}^i x_\beta{}^k g_{ik}$, $x_\alpha{}^i = \partial x^i/\partial u^\alpha$, where $x^i(u^\alpha)$ is a parametric representation of the hypersurface (sphere) $F(x) = 1/\sqrt{K} = \text{const.}$, $K > 0$. Obviously $\gamma_{\alpha\beta}$ may be interpreted as the Riemann metric induced in the sphere by the Riemann metric (3) of the embedding space. Hence Varga's results concerning the curvature of this sphere may easily be regained by standard argumentations on subspaces of Riemann spaces. (See [7], and for the infinite-dimensional case to which we may as well extend Varga's theorems, see [3, p. 143].) We apply the general definition to the calculation of the second fundamental form $l_{\alpha\beta}$ of the hypersurfaces:

$$\begin{aligned} l_{\alpha\beta} = x_\alpha{}^i x_\beta{}^k \nabla_k n_i &= x_\alpha{}^i x_\beta{}^k (\partial_k n_i - \Gamma_{ik}{}^r n_r) = -Kx_\alpha{}^i x_\beta{}^k (g_{ir}\partial_k x^r) \\ &= -\sqrt{K} x_\alpha{}^i x_\beta{}^k g_{ik} = -\sqrt{K} g_{\alpha\beta}, \end{aligned}$$

where Ricci's lemma, equation (3), and the normal vector $n^i = x^i \cdot K$. $n_i = g_{ik}x^k$ have been used. From Gauss' equation [7, p. 242] we obtain for the curvature of the sphere

$$R_{\alpha\beta\gamma\delta} = x_\alpha{}^i x_\beta{}^j x_\gamma{}^k x_\delta{}^l R_{ijkl} - K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$$

which implies Varga's theorems.

6. A theorem on geodesics. By (6) the differential equation for the geodesics of the Riemann metric (4) is

$$\ddot{x}^i + \tfrac{1}{2} g^{ir}(x) g_{rlm}(x) \dot{x}^l \dot{x}^m = 0. \tag{10}$$

Obviously the straight lines through the origin $x^i(t) = t \cdot y^i$ are geodesics. Now the following question arises: What are the conditions for $\mathfrak{C}$ (or g) which imply that all straight lines (not only those passing through 0) are geodesics? The answer is:

6.1. *The system of geodesics of* (4) *coincides with the system of all straight lines if and only if* $\mathfrak{C}$ *is an ellipsoid having* 0 *as its center, or, in other words,* $g_{ik} = \text{const.}$

Proof. Obviously this condition is sufficient. We shall first prove, that it is necessary for $n = 2$. By a well known theorem [7, p. 287] a necessary and sufficient condition for the geodesics to be straight lines (in a linear coordinate system) is $\Gamma_{kj}^i = \delta_k^i \phi_j + \delta_j^i \phi_k$, or, for any base,

$$\Gamma_{22}^1 = \Gamma_{11}^2 = 2\Gamma_{12}^1 - \Gamma_{22}^2 = 2\Gamma_{12}^2 - \Gamma_{11}^1 = 0.$$

In our case (equation (6)) these four equations may be written

$$\begin{aligned} g^{11} g_{122} + g^{12} g_{222} &= 0 \\ g^{21} g_{111} + g^{22} g_{112} &= 0 \\ 2g^{11} g_{112} + g^{12} g_{122} - g^{22} g_{222} &= 0 \\ -g^{11} g_{111} + g^{12} g_{112} + 2g^{22} g_{122} &= 0. \end{aligned}$$

The determinant of this system of four linear equations for the four unknowns g_{111}, g_{112}, g_{122}, g_{222} is $[(g^{11}g^{22}) - (g^{12})^2]^2 > 0$. Hence the system admits only the solution $g_{ijk} = 0$. That proves 6.1 for the two-dimensional case. For spaces $\mathfrak{B}^1$ of any dimension greater than 2 with the straight lines as geodesics of (4) the argumentation runs as follows. The straight lines are then a fortiori geodesics with regard to the Riemann metric induced in a two dimensional plane passing through the origin which is again of the form considered here. By the above result every two-dimensional subspace is euclidean (or, the indicatrix in it is an ellipse), and from this the euclidean character of the space follows by a well known theorem. This completes the proof of 6.1.

MATHEMATISCHES FORSCHUNGSINSTITUT, OBERWOLFACH, SCHWARZWALD
AND
BARNARD COLLEGE, COLUMBIA UNIVERSITY, NEW YORK, N. Y.

REFERENCES.

[1] E. Cartan, *Les espaces de Finsler*, Paris, 1934.

[2] G. Landsberg, "Krümmungstheorie und Variationsrechnung," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 16 (1907), pp. 547-557.

[3] D. Laugwitz, "Differentialgeometrie ohne Dimensionsaxiom I, II," *Mathematische Zeitschrift*, vol. 61 (1954), pp. 100-118 and 134-149.

[4] ———, "Grundlagen für die Geometrie der unendlichdimensionalen Finslerräume," *Annali di Matematica Pura ed Applicata*, ser. 4, vol. 41 (1956), pp. ———.

[5] E. R. Lorch, "A curvature study of convex bodies in Banach spaces," *ibid.*, ser. 4, vol. 34 (1953), pp. 105-112.

[6] ———, "Differentiable inequalities and the theory of convex bodies," *Transactions of the American Mathematical Society*, vol. 71 (1951), pp. 243-266.

[7] J. A. Schouten, *Ricci calculus*, 2nd ed., Berlin, 1954.

[8] O. Varga, "Die Krümmung der Eichfläche des Minkowskischen Raumes und die geometrische Deutung des einen Krümmungstensors des Finslerschen Raumes," *Abhandlungen aus dem Mathematischen Seminar der Universität* Hamburg, vol. 20 (1955), pp. 41-51.

ADDENDA TO THE PAPER ON BÔCHER'S THEOREM

(vol. 76 (1954), pp. 183-190).*

By AUREL WINTNER.

If $(a_{ik}) = a = a(t)$, where $0 \leqq t < \infty$, is an n by n matrix of continuous functions $a_{ik}(t)$, let $a \in S$ mean that $a(t)$ has the following property: Corresponding to every constant vector c, there exists a solution $x = x(t)$ of $dx/dt = a(t)x$ satisfying $x(t) \to c$ as $t \to \infty$ (in view of the principle of superposition, there is just one $x(t) = x_c(t)$ belonging to the "initial condition" $x(\infty) = c$, if $a \in S$). If $a \in R$ means the conditional integrability of $a(t)$, that is, the existence of a finite limit, as $T \to \infty$, of the integral of $a(t)$ over $0 \leqq t \leqq T$, and if $n > 1$, then $a \in R$ is neither necessary nor sufficient for $a \in S$ (cf. the paper quoted in the title, which will be referred to as *loc. cit.*). On the other hand. if $a \in L$ means (as usual, with $L = L^1$) that $\gamma \in R$ holds for the norm, $\gamma = |a|$, of a also, then, according to a criterion which goes back to Bôcher, $a \in L$ is sufficient for $a \in S$ (for references to the implication $L \Rightarrow S$, and for generalizations of $L \Rightarrow S$ derived from $L \Rightarrow S$ itself, cf. *loc. cit.*).

There is, however, something unsatisfactory in the standard criterion, $L \Rightarrow S$. In fact, this criterion fails to contain the fact that, as readily seen from a quadrature, $a \in R$ is sufficient (and necessary) for $a \in S$ in the scalar case, $n = 1$. It is therefore of interest that $L \Rightarrow S$ can be improved to the following criterion: *If $a_0(t)$ denotes the diagonal matrix the diagonal elements of which are those of $a(t)$, then $a_0 \in R$ and $a - a_0 \in L$ together are sufficient for $a \in S$.* It will be clear from the following proof that (and in which manner) this extended criterion can be combined with the criteria given *loc. cit.*

First, it follows from $L \Rightarrow S$ by a known application of the method of the variation of constants (cf. vol. 68 (1946), p. 200, of this Journal), that if $\beta = \beta(t)$ is any continuous matrix the real part of the trace of which has an indefinite integral posessing a lower bound ($> -\infty$), then $\beta \in S$ and $a - \beta \in L$ suffice for $a \in S$. Since the trace condition is certainly satisfied if $\beta \in R$, it follows, by choosing $\beta = a_0$, that the italicized assertion is proved

---

* Received September 26, 1956.

if it is ascertained that $\alpha_0 \in S$. But $\alpha_0 \in S$ follows from the assumption $\alpha_0 \in R$. In fact, since $\beta \in R$ is sufficient (and incidentally, as mentioned above, necessary as well) for $\beta = S$ in the scalar case, it is clear that the same is true in the case of any diagonal matrix β.

* *
*

If a scalar function $f(t)$ is continuous for $0 \leqq t < \infty$, what kind of its "smallness" (for large t) will ensure that the scalar differential equation $x'' + \{1 + f(t)\}x = 0$ represents an "adiabatic" perturbation of harmonic oscillator $y'' + y = 0$? By this is meant the existence of two solutions, say $x = x_j(t)$, where $j = \pm 1$, satisfying $x_j(t) \sim e^{jit}$ and $x_j'(t) \sim jie^{jit}$ as $t \to \infty$. Let this property of the coefficient function $1 + f(t)$ be symbolized by $f \in P$. Corresponding to the preceding notations, $f \in R$ and $f \in L$ will refer to the integrability and the absolute integrability of f (on $0 \leqq t < \infty$) respectively.

It is well-known that, as a consequence of the standard criterion $L \Rightarrow S$ for an α, the criterion $L \Rightarrow P$ holds for an f (for a direct proof, cf. pp. 71-73 of the London (1931) edition of Weyl's Quantum Mechanics). But the assumption $f \in L$ is quite drastic. It can be greatly improved by having recourse to the refinements of $L \Rightarrow S$ given *loc. cit.* In fact, the following improvement of $L \Rightarrow P$ can be obtained: *For $f \in P$ it is sufficient that each of the three functions f_k, where $k = -1, 0, 1$ and $f_k(t) = f(t)e^{2kti}$, satisfy the following two conditions: $f_k \in R$ and $F_k(t) \in L$, where*

$$F_k(t) = f(t) \int_t^\infty f(s)e^{-2kti}\,ds \qquad \left(\int_t^\infty = \lim_{T\to\infty} \int_t^T\right).$$

The involvement of $f_{\pm 1}(t) = f(t)e^{\pm 2ti}$ (besides $f_0(t) \sim f(t)$ itself), being a *resonance* restriction, is curious, since the problem concerns itself with a *homogeneous* linear differential equation (in this connection cf., however, O. Perron, Math. Ztschr., vol. 32 (1930), p. 705 and Satz 1).

The proof proceeds as follows: It was shown *loc. cit.* that $\alpha \in S$ whenever $\alpha \in R$ and $\gamma \in L$, where $\gamma(t)$ is either of the matrix products $\alpha(t)\alpha^0(t)$, $\alpha^0(t)\alpha(t)$ in which $\alpha^0(t)$ denotes the integral (R) of $\alpha(t)$ over the half-line (t, ∞). On the other hand, it follows by a known application of the method of the variation of constants that $f \in P$ is equivalent to $\alpha \in S$, where $\alpha = \alpha_f$ denotes the following binary matrix:

$$\alpha_f(t) = f(t)\begin{pmatrix} -uv & -v^2 \\ u^2 & uv \end{pmatrix}, \text{ where } u = \cos t,\ v = \sin t$$

(cf., e.g., vol. 69 (1947), pp. 262-263 of this Journal, where the corresponding relation is calculated for the generalization $y''+g(t)y=0$ of the present $y''+y=0$). But it is readily seen that, for this $a(t)=a_f(t)$, the two matrix conditions $a\in R$, $aa^0\in L$ can be reduced to three pairs $f_k\in R$, $F_k\in L$ of scalar conditions.

An instructive illustration results by choosing $f(t)=(\sin at)/t$, where a is a real constant $\neq 0$. Then the standard sufficient condition, $f\in L$, for $f\in P$ is violated for every a. But it is clear that all $3+3$ conditions $f_k\in R$, $F_k\in L$ (with $F_k(t)=O(t^{-2})$ as $t\to\infty$) are satisfied unless $a=\pm 2$. Consequently, $f\in P$ *if* $f(t)=(\sin at)/t$, *provided that* $a\neq\pm 2$. This is the more interesting as the proviso, taking care of the "resonance effect" referred to above, is known to be indispensable. Cf. vol. 69 (1947), p. 269 of this Journal, where $a=2$, and, for a result which claims much less (in fact, just the boundedness of the solutions) than the last italicized result if $a\neq\pm 2$, G. Ascoli, Rend. Acc. Naz. Lincei, ser. 8, vol. 9 (1950), pp. 210-213. Ascoli's corresponding general statements (p. 131) are based on the assumption that boundedness is the same thing as asymptotic equivalence, an assumption which, however, is not in general satisfied.

It may finally be mentioned that if the $3+3$ conditions of the preceding general theorem are satisfied by an $f(t)$, then it is seen, by a repetition of the preceding proof for the asymptotic equivalence with $y''+y=0$, that *the general criterion remains true if the preceding* $x''+\{1+f(t)\}x=0$ *is replaced by* $x''+f(t)x'+x=0$.

THE JOHNS HOPKINS UNIVERSITY.

CORRESPONDENCE.

A correspondent, who wishes to remain anonymous, writes as follows:

SIR,

In a paper which has been published by your Journal (October, 1949), W. L. Chow proved the following theorem:

Every closed analytic subvariety V of the projective space $\boldsymbol{P}_r(\boldsymbol{C})$ is algebraic.

I should like to point out that there is a very simple proof for this *if V is assumed to be free from singularities.*

It is no restriction to assume that V is connected; let n be its complex dimension. In the polynomial ring $\boldsymbol{C}[X_0, \cdots, X_r]$, the homogeneous polynomials which vanish on V generate a homogeneous ideal $\mathfrak{p}$. Since V is connected, $\mathfrak{p}$ is a prime ideal and defines therefore an irreducible algebraic variety W, which is the smallest one containing V; call m its dimension; we have $m \geqq n$. Every rational function f on W can be written as $f = P/Q$, where P, Q are homogeneous polynomials of the same degree and Q is not in $\mathfrak{p}$; therefore f induces a meromorphic function on V. This shows that the field L of rational functions on W is isomorphic to a subfield of the field K of meromorphic functions on V. It is well-known (cf., e.g., C. L. Siegel's proof, Göttingen Nachrichten 1955) that K has at most the degree of transcendency n; so we get $m \leqq n$, and hence $m = n$; therefore W has the same dimension as V. It is also well-known that W is analytically irreducible (this follows from the fact that the set of all simple points on W is connected, which is easily proved by induction on the dimension, using suitable hyperplane sections). Therefore $W = V$.

Yours, etc.

X. X. X.

CORRECTIONS TO "ENGEL RINGS AND A RESULT OF HERSTEIN AND KAPLANSKY."*

By M. P. Drazin.

I. N. Herstein has pointed out to the author that the proof of Theorem 2.1 in the paper named above (this *Journal*, vol. 77 (1955), pp. 895-913) is invalidated by the use of an incorrect expression for the typical element d of the ideal generated by x^m (the theorem may nevertheless be true, but must now be regarded as unproven). This in turn invalidates the proofs given for Lemma 4.1 (i) and Theorems 3.1, 4.1 and 6.2 (i.e. all those results depending on the explicit use of the Koethe radical).

These deletions involve consequential modifications in Theorems 5.1 and 6.1. In Theorem 5.1, we must disallow the maximal alternative, and the proof in the minimal case must be based on Theorem 4.5 rather than on Theorem 4.1. In Theorem 6.1, the final clause "in particular, by Theorem 4.1, R is commutator-nil" should be deleted. Also, in the proof of Theorem 5.4, the second sentence must be based on A. S. Amitsur's paper [1] ("On rings with identities," *Journal of the London Mathematical Society*, vol. 30 (1955), pp. 464-470) rather than on Theorem 4.1.

Further, as already stated (this *Journal*, vol. 78 (1956), p. 224), Theorem 6.5 can no longer be upheld, and, in place of Theorem 6.4, we can substantiate only the following weaker assertion:

If a division ring D has the weak K-property, i.e. if there correspond to each pair $x, y \in D$ integers $k = k(x,y)$, $m = m(x,y)$ and $n = n(x,y)$ such that $e_k(x^m, y^n) = 0$, then, if we modify n appropriately, we may always replace k by 1 while keeping m as given; if D has characteristic zero then in fact no modification of n is needed, i.e. $[x^{m(x,y)}, y^{n(x,y)}] = 0$ for all x, y.

It goes without saying that many of the linking passages and informal remarks in the paper under correction must now be looked on as (at best) misleading, but we shall not discuss these here. However, the remaining formally enunciated results, and in particular the key Theorems 4.3, 4.4 and 4.5, are indeed true as originally stated.

Trinity College, Cambridge, England.

* Received August 27, 1956.

ERRATA.

S. Abhyankar, "On the ramification of algebraic functions," this JOURNAL, vol. 77 (1955), pp. 575-592.

(1) Page 580 line 6 in the proof of Theorem 1: Instead of "P^* is at finite distance" read "$g_i \varepsilon M^*$ for $i = 1, 2, \cdots, m$."

(2) Page 581 line 6: Instead of "Y_{m-r}" read "Y_{m+r}."

(3) Page 581 lines 7 and 8: Instead of "$i = m+1, m+2, \cdots, m+r$," read "$i = 1, 2, \cdots, m$."

(4) Page 581 line 26: Instead of "G as a point in $S_{m+1}/k(x_1, x_2, \cdots, x_{r-1})$," read "$U^*$ as a point in $S_{m+1}/k(x_2, x_3, \cdots, x_r)$."

(5) Page 581 lines 27 and 28: Twice instead of "at $Y_1 = y_1$, $Y_2 = y_2$, $\cdots$, $Y_{m+r} = y_{m+r}$" read "at U^*."

ON THE PREPARATION OF MANUSCRIPTS.

The following instructions are suggested or dictated by the necessities of the technical production of the American Journal of Mathematics. Authors are urged to comply with these instructions, which have been prepared in their interests.

Manuscripts not complying with the standards usually have to be returned to the authors for typographic explanation or revisions and the resulting delay often necessitates the deferment of the publication of the paper to a later issue of the Journal.

Horizontal fraction signs should be avoided. Instead of them use either solidus signs / or negative exponents.

Neither a solidus nor a negative exponent is needed in the symbols $\frac{1}{2}, \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{2\pi i}$, which are available in regular size type.

Binomial coefficients should be denoted by C_k^n and not by parentheses. Correspondingly, for symbols of the type of a quadratic residue character the use of some non-vertical arrangement is usually imperative.

Use the exponent $\frac{1}{2}$ instead of the square root sign.

Replace $e^{(\ )}$ by exp( ) if the expression in the parenthesis is complicated.

By an appropriate choice of notations, avoid unnecessary displays.

Simple formulae, such as $A + iB = \frac{1}{2}C^*$ or $s_n = a_1 + \cdots + a_n$, should not be displayed (unless they need a formula number).

Use $'$ or $d\ /dx$, possibly D, but preferably not a dot, in order to denote ordinary differentiation and, as far as possible, a subscript in order to denote partial differentiation (when the symbol ∂ cannot be avoided, it should be used as $\partial\ /\partial x$).

Commas between indices are usually superfluous and should be avoided if possible.

In a determinant use a notation which reduces it to the form $\det a_{ik}$.

Subscripts and superscripts cannot be printed in the same vertical column, hence the manuscript should be clear on whether a_i^k or a_i^k is preferred. (Correspondingly, the limits of summation must not be typed *after* the Σ-sign, unless either Σ_1^m or $\Sigma^m{}_1$ is desired.) If a letter carrying a subscript has a prime, indicate whether x_i' or x_i' is desired.

Experience shows that a tilde or anything else *over* a letter is very unsatisfactory. Such symbols often drop out of the type after proof-reading and, when they do not, they usually appear uneven in print. For these reasons we advise against their use. This advice applies also to a bar over a Greek or German letter (for the symbol of complex conjugation an asterisk is often allowed by the context). Type carrying bars over ordinary size italic letters of the Latin alphabet is available.

Bars reaching over several letters should in any case be avoided (in particular, type lim sup and lim inf instead of lim with upper and lower bars).

Repeated subscripts and superscripts should be used only when they cannot be avoided, since the index of the principal index usually appears about as large as the principal index. Bars and other devices *over* indices cannot be supplied. On the other hand, an asterisk or a prime (to be printed *after* the subscript) is possible on a subscript. The same holds true for superscripts.

Distinguish carefully between l. c. "oh," cap. "oh" and zero. One way of distinguishing them is by underlining one or two of them in different colors and explaining the meaning of the colors.

Distinguish between ϵ (epsilon) and $\in$ or ε (symbol), between x (eks) and $\times$ (multiplication sign), between l. c. and cap. phi, between l. c. and cap. psi, between l. c. k and kappa and between "ell" and "one" (for the latter, use ł and 1 respectively).

Avoid unnecessary footnotes. For instance, references can be incorporated into the text (parenthetically, when necessary) by quoting the number in the bibliographic list, which appears at the end of the paper. Thus: "[3], pp. 261-266."

Except when informality in referring to papers or books is called for by the context, the following form is preferred:

[3] O. K. Blank, "Zur Theorie des Untermengenraumes der abstrakten Leermenge," *Bulletin de la Société Philharmonique de Zanzibar*, vol. 26 (1891), pp. 242-270.

In any case, the references should be precise, unambiguous and intelligible.

Usually sections numbers and section titles are printed in bold face, the titles "Theorem," "Lemma" and "Corollary" are in caps and small caps, "Proof," "Remark" and "Definition" are in italics. This (or a corresponding preference) should be marked in the manuscript. Use a period, and *not* a colon, after the titles Theorem, Lemma, etc.

German, script and bold face letters should be underlined in various colors and the meaning of the colors should be explained. The same device is needed for Greek letters if there is a chance of ambiguity. In general, mark all cap. Greek letters.

All instructions and explanations for the printer can conveniently be collected on a separate sheet, to be attached to the manuscript.

In case of doubt, recent issues of the Journal may be consulted.

THE JOHNS HOPKINS PRESS · · BALTIMORE 18

American Journal of Hygiene. Edited by D. BODIAN, Managing Editor, A. M. BAETJER, R. M. HERRIOTT, M. MERRELL, P. E. SARTWELL, and T. B. TURNER. Publishing two volumes of three numbers each year, volume 63 is now in progress. Subscription $12 per year. (Foreign postage, 50 cents; Canadian postage, 25 cents.)

American Journal of Mathematics. Edited by A. WEIL and A. WINTNER. Quarterly. 8 vo. Volume 78 in progress. $8.50 per volume. (Foreign postage, 50 cents; Canadian, 25 cents.)

American Journal of Philology. Edited by H. T. ROWELL, LUDWIG EDELSTEIN, K. MALONE, B. D. MERITT, JAMES H. OLIVER, and EVELYN H. CLIFT, Secretary. Volume 77 is in progress. $6.00 per volume. (Foreign postage, 50 cents; Canadian 25 cents.)

Bulletin of the History of Medicine. OWSEI TEMKIN, Editor. Bi-monthly. Volume 30 in progress. 8vo. Subscription $6 per year. (Foreign postage, 50 cents; Canadian 25 cents.)

Bulletin of the Johns Hopkins Hospital. Edited by PHILIP F. WAGLEY. Monthly. 8vo. Subscription $8 per year. Vol. 98-99 is in progress. (Foreign postage, 50 cents; Canadian, 25 cents.)

ELH. A Journal of English Literary History. Edited by E. T. NORRIS (Managing Editor) and D. C. ALLEN, G. E. BENTLEY, F. MICHAEL KROUSE, C. P. LYONS, J. HILLIS MILLER, ROY H. PEARCE and E. R. WASSERMAN. Quarterly 8vo. Volume 23 in progress. $4.00 per volume. (Foreign postage, 40 cents; Canadian, 20 cents.)

Johns Hopkins Studies in Romance Literatures and Languages. †H. C. LANCASTER, Editor. 8vo. Seventy-six numbers have been published.

Johns Hopkins University Studies in Archaeology. DAVID M. ROBINSON, Editor. 8vo. Thirty-nine volumes have appeared.

Johns Hopkins University Studies in Geology. JOSEPH T. SINGEWALD, JR., Editor. 8vo. Sixteen numbers have been published.

Johns Hopkins University Studies in Historical and Political Science. Under the direction of the Departments of History, Political Economy and Political Science. 8vo. Volume 72 in progress.

Modern Language Notes. Edited by KEMP MALONE, DON CAMERON ALLEN, CHARLES R. ANDERSON, ANNA HATCHER, WILLIAM H. MCCLAIN, GEORGES POULET, BRUCE W. WARDROPPER, EARL R. WASSERMAN, and S. K. HENINGER, JR., *Secretary*. Eight times yearly. 8vo. Volume 71 in progress. $6 per volume. Foreign postage, 50 cents; Canadian, 25 cents.)

A complete list of publications will be sent on request

Lectures given at the Johns Hopkins University
(spring term, 1953) by

CARL LUDWIG SIEGEL

TOPICS IN CELESTIAL MECHANICS

Mimeographed. Bound. ii + 237 pages. Baltimore 1955.

Price $3.50 postpaid. Orders should be sent and checks made payable to

THE JOHNS HOPKINS BOOKSTORE, BALTIMORE 18, MD.